EUCLIDES ELEMENTS;

The whole Fifteen Books compendiously Demonstrated

By Mr. ISAAC BARROW Fellow of *Trinity Colledge* in CAMBRIDGE.

And Translated out of the Latin.

Καθαρμοὶ ψυχῆς λογικῆς εἰσιν αἱ μαθηματικαὶ ἐπιστῆμαι. Hierocl.

LONDON,
Printed by R. DANIEL, for WILLIAM NEALAND Bookseller in *Cambridge*; and are to be sold there, and at the *Crown* in *Duck-Lane*, 1660.

THE AUTHOR'S PREFACE.

IN order to the Reader's ſatisfaction concerning the Book put into his hand, I am to advertiſe him of ſome few things, and that according to the nature of the Work, briefly; as followeth. My Undertaking aimed principally at two Ends. The firſt of which was to conjoin the greateſt Compendiouſneſs of Demonſtration with as much Perſpicuity as the quality of the ſubject would admit; that ſo the Volume might bear no bigger bulk then would render it conveniently portable. Which I have ſo farr attained, that though poſſibly ſome other perſon might with greater curioſity, yet (I preſume) none could with more conciſeneſſe have demonſtrated moſt propoſitions; eſpecially, ſince I have altered nothing in the number and order of the Propoſitions, nor taken the liberty to leave out any one of *Euclide's* as leſſe neceſſary, or to reduce certain of the eaſieſt into the Claſſis of Axiomes. Which notwithſtanding ſome have done; as that moſt accurate Geometrician *Andr. Tacquet*, whom I mention the rather, becauſe I eſteem it ingenuous to acknowledge ſome things taken from him. And, indeed, I ſhould have attempted nothing after his moſt elegant Edition, had it not pleaſed that learned Perſon to publiſh

only Eight of *Euclide's* Books illustrated by his paines, either slighting or undervaluing the other Seven as lesse relating to the Elements of Geometry. But I had a different Purpose from the beginning; not to compose Elements of Geometry any-wise at my discretion, but to demonstrate *Euclide* himself, and all of him, and that with all possible brevity. For as for Foure of his Books, the Seventh, Eighth, Ninth and Tenth, although they do not so neerly pertain to the Elements of Plane and Solid Geometry, as the Six First & the Two subsequent; yet no man that ha's arriv'd to any measure of skill in Geometry is ignorant how exceedingly usefull they are in Geometricall matters, aswell in regard of the very neer alliance between Arithmetick and Geometrie, as for the knowledge of Commensurable and Incommensurable Magnitudes which is highly important to the understanding both of Plane and Solid Figures. And the noble Theory of the Five Regular Bodies, contained in the Three Last Books, could not be omitted without prejudice & injury; since our Author of these Elements, being a Sectator of *Plato's* Schole, is reported to have compiled the whole Systeme only in reference to that Contemplation; which *Proclus* attesteth in these words, Ὅθεν δὴ καὶ τῆς συμπάσης στοιχειώσεως τέλος προεστήσατο τὴν τῶν καλουμένων Πλατωνικῶν σχημάτων σύστασιν. Moreover, I was easily induc'd to believe, that it would be acceptable to all Lovers of these Sciences to have the Intire work of *Euclide* by them, as it is usually cited and recommended by all men. Wherefore I determin'd

min'd to leave out no Book or Proposition of those which are found in *Peter Herigon*, whose footsteps I became necessitated to follow closely by having resolved to make use for the most part of the Schemes of his Book, upon a foresight that my speedy departure out of *England*, would not allow me time to describe New, although I sometimes desired so to doe. Upon the same account also I purposed to use generally no other then *Euclide's* own Demonstrations, contracted into a more succinct form, saving perchance in the Second and Thirteenth, & sparingly in the Seventh, Eighth, and Ninth Books, where it seem'd convenient to vary something from him. So that it may be reasonably hoped that in this Particular our own Design and the Wishes of the Studious are in some manner satisfi'd.

The other End aimed at, was in favour of Their desires who more affect Symbolical then Verbal Demonstrations. In which kind, seeing most of our own Nation are accustomed to the Notes of Mr. *Oughtred*, I esteemed it more convenient to make use of them principally throughout. For no man hitherto that I know of, saving only *Peter Herigon*, ha's attempted to set forth and interpret *Euclide* according to this way. The Method of which most learned Person, though in many other respects very excellent, and exactly accommodated to his peculiar purpose, seem'd to me notwithstanding doubly defective. First, in that, whereas of severall Propositions brought to the proving of some one Theoreme or Probleme the Latter

do's

do's not alwaies depend on the Former, yet when they do cohere one with another, and when not, cannot readily enough be known, either from their order or any other way; whence it not seldome comes to passe, that through the want of Conjunctions and Adjectives, *Ergo, rursus*, &c. there arises difficulty and occasion of doubting, especially to such as are but little vers'd therein. And in the next place, it oftentimes falls out that the said Method cannot avoid superfluous repetitions, whereby the Demonstrations become sometimes prolixe, and sometimes perplex'd and intricate. All which Inconveniences are easily remedied in our Way by the intermingling of Words and Signes at discretion. And thus much may suffice to be premised concerning the Intent and Method of this Compendium. I shall not alledge in favour of my self the scantnesse of time allotted to this Work, nor the avocations of affairs, nor the scarcity of Helps to this sort of Studies amongst us (as I might not untruly) out of fear lest my Performance should not throughly please every body: But I wholly submit to the faire Censure and Judgement of the Ingenuous Reader, what I have undertaken for the advantage of his Studies; to be approved, if he find it serviceable thereunto; or, if otherwise, rejected.

Ad amicissimum Virum, I. C. de EUCLIDE contracto, Εὐφημισμός.

FActum bene! didicit Laconice loqui
Senex profundus, & aphorismos induit.
Immensa dudum margo commentarii
Diagramma circuit minutum; utque Insula
Problema breve natabat in vasto mari.
Sed unda jam detumuit; & glossa arctior
Stringit Theoremata: minoris anguli
Lateribus ecce totus Euclides *jacet,*
Inclusus olim velut Homerus in nuce;
Pluteoque sarcina modo qui incubuit, levis
En fit manipulus. Pelle in exigua latet
Ingens Mathesis, matris utero Hercules,
In glande quercus, vel Ithaca Eurus in pila.
Nec mole dum decrescit, usu fit minor;
Quin auctior jam evadit, & cumulatius
Contracta prodest erudita pagina.
Sic ubere magis liquor è presso effluit;
Sic pleniori vasa inundat sanguinis
Torrente cordis Systole; sic fusius
Procurrit æquor ex Abylæ angustiis.
Tantilli operis ars tanta referenda unice est
BAROVIANO *nomini, ac solertiæ.*
Sublimis euge mentis ingenium potens!
Cui invium nil, arduum esse nil soles;
Sic usque pergas prospero conamine,
Radiusque multum debeat ac abacus tibi;
Sic crescat indies feracior seges,
Simili colonum germine assiduo beans.
Specimen futuræ messis hic siet labor,
Magnæque famæ illustria hæc præludia.
Iuvenis dedit qui tanta, quid dabit senex?

Car. Robotham, *CANTAB.*
Coll. Trin. Sen. Soc.

The Explication of the Signes or Characters.

$=$	Signifies	Equall.
$\sqsubset$	Signifies	Greater.
$\sqsupset$	Signifies	Lesser.
$+$	Signifies	More, or to be added.
$-$	Signifies	Lesse, or to be subtracted.
$-:$	Signifies	The Difference, or Excesse; Also, that all the quantities which follow, are to be subtracted, the Signes not being changed.
x	Signifies	Multiplication, or the Drawing one side of a Rectangle into another. The same is denoted by the Conjunction of letters; as $AB = A \times B$.
$\surd$	Signifies	The Side or Root of a Square, or Cube, &c.
Q & q	Signifies	A Square.
C & c	Signifies	A Cube.
Q.Q.	Signifies	The ratio of a square number to a square number.

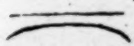
Other Abbreviations of words, where ever they occurr, the Reader will without trouble understand of himself; saving some few, which, being of lesse generall use, we referr to be explained in their own places.

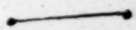
THE

THE FIRST BOOK OF EUCLIDE'S ELEMENTS.

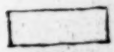
Definitions.

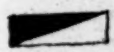
I. A Point is that which hath no part. 

II. A Line is a longitude without latitude. 

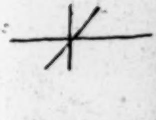
III. The ends, or limits, of a line are points. 

IV. A right line is that which lyes equally betwixt it's points.

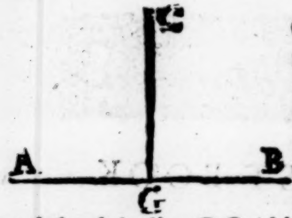
V. A Superficies is that which hath only longitude and latitude. 

VI. The extremes, or limits, of a superficies are lines. 

VII. A plaine superficies is that which lyes equally betwixt it's lines.

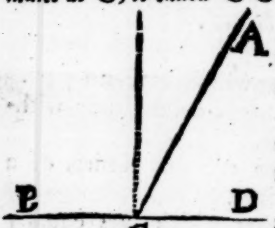
VIII. A plaine Angle is the inclination of two lines the one to the other, the one touching the other in the same plain, yet not lying in the same strait line. 

IX. And if the lines which contein the angle be right lines, it is called a right-lined angle.



X. When a right line C G standing upon a right line A B, makes the angles on either side thereof, C G A, C G B, equall one to the other, then both those equall angles are right angles; and the right line C G, which standeth on the other, is termed a Perpendicular to that (A B) whereon it standeth.

Note. *When severall angles meet at the same point (as at* G) *each particular angle is described by three letters; whereof the middle letter sheweth the angular point, and the two other letters the lines that make that angle: As the angle which the right lines* C G, A G *make at* G, *is called* C G A, *or* A G C.



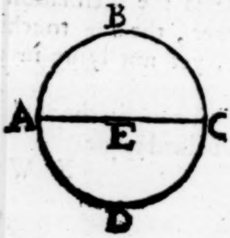
XI. An obtuse angle is that which is greater then a right angle; as A C B.

XII. An acute angle is that which is lesse then a right angle; as A C D.

XIII. A Limit, or Term, is the end of any thing.

XIV. A Figure is that which is conteined under one or more terms.

XV. A Circle is a plain figure conteined under one line, which is called a Circumference; unto which all lines drawn from one point within the figure, and falling upon the circumference thereof, are equall the one to the other.



XVI. And that point is called the Centre of the circle.

XVII. A Diameter of a circle is a right line drawn through the centre thereof, and ending at the circumference on either

ther side, dividing the circle into two equall parts.

XVIII. A Semicircle is a figure which is conteined under the diameter and under that part of the circumference which is cut off by the diameter.

In the circle E A B C D, E *is the centre*, A C *the diameter*, A B C *the semicircle.*

XIX. Right-lined figures are such as are conteined under right lines.

XX. Three-sided or Trilateral figures are such as are conteined under three right lines.

XXI. Four-sided or Quadrilateral figures are such as are conteined under four right lines.

XXII. Many-sided figures are such as are conteined under more right lines then four.

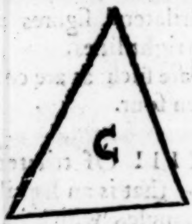
XXIII. Of trilateral figures, that is an Equilateral Triangle, which hath three equal sides; as the Triangle A.

XXIV. Isosceles is a Triangle which hath onely two sides equall; as the Triangle B.

XXV. Scalenum is a Triangle whose three sides are all unequall; as C.

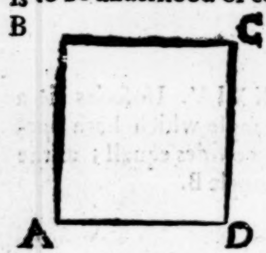
XXVI. Of these trilateral figures, a Right-angled Triangle is that which hath one right angle; as the Triangle A.

XXVII. An Amblygonium, or obtuse-angled Triangle, is that which hath one angle obtuse; as B.



XXVIII. An Oxygonium, or acute-angled Triangle, is that which hath three acute angles; as C.

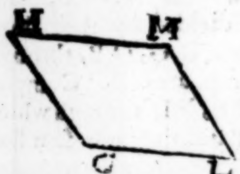
An Equiangular, or equall-angled, figure is that whereof all the angles are equall. Two figures are equiangular, if the severall angles of the one figure be equall to the severall angles of the other. The same is to be understood of equilateral figures.



XXIX. Of quadrilateral, or four-sided, figures, a Square is that whose sides are equall, & angles right; as A B C D.

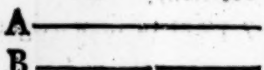
XXX. A figure on the one part longer, or a long square, is that which hath right angles, but not equall sides; as A B C D.

XXXI. A Rhombus, or diamond-figure, is that which hath four equall ſides, but is not right-angled; as A.



XXXII. A Rhomboides, or diamond-like figure, is that whoſe oppoſite ſides, and oppoſite angles, are equall; but hath neither equall ſides nor right angles; as GLMH.

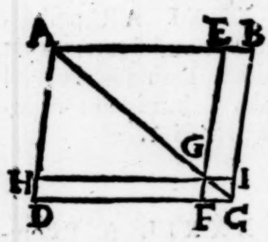
XXXIII. All other quadrilateral figures beſides theſe are called *Trapezia*, or Tables; as GNDH.



XXXIV. Parallel, or equidiſtant, right lines are ſuch, which being in the ſame ſuperficies, if infinitely produced, would never meet; as A and B.

XXXV. A Parellelogram is a quadrilateral figure, whoſe oppoſite ſides are parallel, or equidiſtant; as GLHM.

 XXXVI. In



XXXVI. In a parallelogram A B C D, when a diameter A C, and two lines E F, H I parallel to the sides, cutting the diameter in one and the same point G, are drawn, so that the parallelogram be divided by them into four parallelograms; those two, D G, G B, through which the diameter passeth not, are called Complements; and the other two, H E, F I, through which the diameter passeth, the Parallelograms standing about the diameter.

A Probleme *is, when something is proposed to be Done or effected.*

A Theoreme *is, when something is proposed to be Demonstrated.*

A Corollary *is a consectary, or some consequent truth gained from a preceding demonstration.*

A Lemma *is the demonstration of some premise, whereby the proof of the thing in hand becomes the shorter.*

Postulates or Petitions.

1. FRom any point to any point to draw a right line.

2. To produce a right line finite, strait forth continually.

3. Upon any centre, and at any distance, to describe a circle.

Axiomes.

1. THings equall to the same third, are also equall one to the other;

As A=B=C. Therefore A=C. Or therefore all, A, B, C, are equall the one to the other.

Note. *When severall quantities are joyned the one to the other continually with this mark =, the first quantity is by vertue of this axiome equall to the last, & every one to every one: In which case we often abstein*

from

from citing the axiome, for brevities ſake; although the force of the conſequence depend thereon.

2. If to equall things you adde equall things, the wholes ſhall be equall.

3. If from equall things you take away equall things, the things remaining will be equall.

4. If to unequall things you adde equall things, the wholes will be unequall.

5. If from unequall things you take away equall things, the remainders will be unequall.

6. Things which are double to the ſame third, or to equall things, are equall one to the other. Underſtand the ſame of triple, quadruple, &c.

7. Things which are half of one and the ſame thing, or of things equall, are equall the one to the other. Conceive the ſame of ſubtriple, ſubquadruple, &c.

8. Things which agree together, are equall one to the other.

The converſe of this axiome is true in right lines and angles, but not in figures, unleſſe they be like.

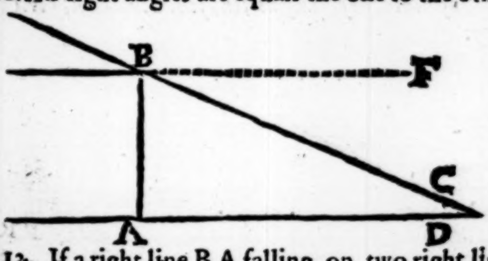
Moreover, magnitudes are ſaid to agree, when the parts of the one being applyed to the parts of the other, they fill up an equall or the ſame place.

9. Every whole is greater then it's part.

10. Two right lines cannot have one and the ſame ſegment (or part) common to them both.

11. Two right lines meeting in the ſame point, if they be both produced, they ſhall neceſſarily cut one the other in that point.

12. All right angles are equall the one to the other.



13. If a right line B A falling on two right lines

CB, make the internall angles on the same side, BAD, ABC, lesse then two right angles, those two right lines produced shall meet on that side, where the angles are lesse then two right angles.

14. Two right lines do not contein a space.

15. If to equall things you adde things unequall, the excesse of the wholes shall be equall to the excesse of the additions.

16. If to unequall things equall be added, the excesse of the wholes shall be equall to the excesse of those which were at first.

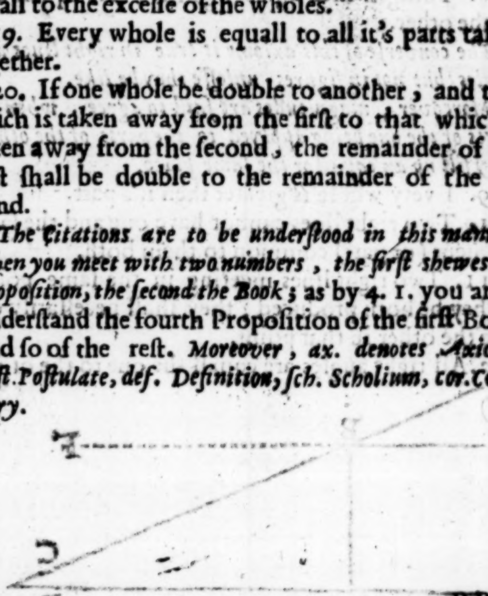
17. If from equall things unequall things be taken away, the excesse of the remainders shall be equall to the excesse of what was taken away.

18. If from things unequall things equall be taken away, the excesse of the remainders shall be equall to the excesse of the wholes.

19. Every whole is equall to all it's parts taken together.

20. If one whole be double to another, and that which is taken away from the first to that which is taken away from the second, the remainder of the first shall be double to the remainder of the second.

The Citations are to be understood in this manner; When you meet with two numbers, the first shewes the Proposition, the second the Book; as by 4. 1. you are to understand the fourth Proposition of the first Book; and so of the rest. *Moreover, ax. denotes Axiome, post. Postulate, def. Definition, sch. Scholium, cor. Corollary.*



PRO-

PROPOSITION I.

Upon a finite right line given A B, *to describe an equilateral triangle* ACB.

From the centres A and B, at the distance of AB, or B A, *a* describe two circles a 3. post.
cutting each other in the point C; from whence *b* draw two right lines CA, b 1. post.
CB. Then is A C *c* = A B *c* = B C *d* = A C. c 15. def. d 1. ax.
e Wherefore the Triangle A C B is equilateral. e 23. def.
Which was to be Done.

Scholium.

After the same manner upon the line A B may be described an Isosceles triangle, if the distances of the equall circles be taken greater or lesse then the line A B.

PROP. II.

At a point given A, *to make a right line* A G *equall to a right line given* B C.

From the centre C, at the distance of C B, *a* de- a 3. post.
scribe the circle C B E, *b* join A C; upon which b 1. post.
c raise the equilateral triangle A D C. *d* Produce c 1. 1.
D C to E. From the centre D, at the distance of D E, d 2. post.
de-

e 3.post. describe the circle DEH; & let DA *e* be produced to the point G in the circumference thereof. Then AG = CB.

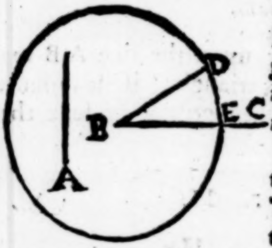
f 15.def. g constr. h 3. ax. k 15 def. l 1.ax. For DG *f* = DE, and DA *g* = DC. Wherefore AG *h* = CE *k* = BC *l* = AG. *Which was to be Done.*

The putting of the point A within or without the line BC varies the cases; but the construction, and the demonstration, are every where alike.

Schol.

The line AG might be taken with a pair of compasses; but the so doing answers to no postulate, as *Proclus* well intimates.

PROP. III.

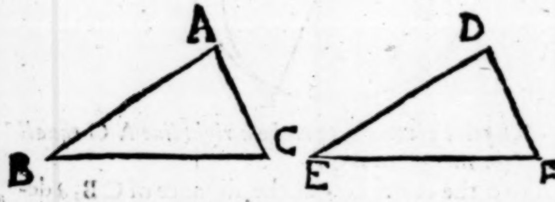


Two right lines, A and BC, being given, from the greater BC to take away the right line BE equall to the lesser A.

a 3. 1. At the point B *a* draw the right line BD = A. The circle described from the center B at the distance of BD shall cut off BE *b* = BD *c* = A *d* = BE. *Which was to be Done.*

b 15.def. c constr. d 1.ax.

PROP. IV.



If two triangles BAC, EDF, *have two sides of the one* BA, AC *equall to two sides of the other* ED, DF, *each to it's correspondent side (that is,* BA = ED, *and*

and A C = D E) *and have the angle* A *equall to the angle* D *conteined under the equall right lines; they shall have the base* B C *equall to the base* E F; *and the triangle* B A C *shall be equall to the triangle* E D F; *and the remaining angles* B, C, *shall be equall to the remaining angles* E, F, *each to each, under which the equall sides are subtended.*

If the point D be applyed to the point A, and the right line D E plac'd upon the right line A B, the point E shall fall upon B, because D E *a* = A B. also the right line D F shall fall upon A C, because the angle A *a* = D. moreover, the point F shall fall upon the point C, because A C *a* = D F. Therefore the right lines EF, BC shall *b* agree, because they have the same termes, & so consequently are equall. Wherefore the triangles B A C, D E F, and the angles B, E, as also the angles C, F, do agree, and are equall. *Which was to be Demonstrated.*

a hyp.
b 14. ax.

P R O P. V.

The angles A B C, A C B, *at the base of an Isosceles triangle* A B C, *are equall one to the other: And if the equall sides* A B, A C *be produc'd, the angles* C B D, B C E, *under the base, shall be equall one to the other.*

a Take A E = A D; and *b* join C D, and B E.

Because, in the triangles ACD, A B E, are A B *c* = A C, and A E *d* = A D, and the angle A common to them both, *e* therefore is the angle ABE = ACD, and the angle AEB *e* = ADC, and the base BE *e* = CD; also EC *f* = DB. Therefore in the triangles B E C, B D C *g* shall be the angle ECB = DBC. *Which was to be Dem.* Also therefore the angle EBC = DCB. but the angle A B E *h* = A C D; therefore the angle A B C *k* = A C B. *Which was to be Dem.*

a 3. 1.
b 1 post.
c hyp.
d constr.
e 4. 1.
f 3. ax.
g 4. 1.
h before.
k 3. ax.

Co-

Coroll.

Hence, Every equilateral triangle is also equiangular.

P R O P. VI.

If two angles ABC, ACB *of a triangle* ABC *be equall the one to the other, the sides* AB, AC *subtended under the equal angles, shall also be equall one to the other.*

If the sides be not equall, let one
a 3. 1. be bigger then the other, suppose BA ⊏ CA. *a* Make
b 1. post. BD = CA, and *b* draw the line CD.

In the triangles DBC, ACB, because BD *c* =
c suppos. CA, and the side BC is common, & the angle DBC
d hyp. *d* = ACB, the triangles DBC, ACB *e* shall be equall
e 4. 1. the one to the other, a part to the whole. *f Which*
f 9 ax. *is impossible.*

Coroll.

Hence, Every equiangular triangle is also equilateral.

P R O P. VII.

Upon the same right line AB *two right lines being drawn* AC, BC, *two other right lines equall to the former,* AD, BD, *each to each (viz.* AD = AC, *and* BD = BC) *cannot be drawn from the same points* A, B, *on the same side* C, *to severall points, as* C *and* D, *but onely to* C.

1. *Case.* If the point D be set in the line AC, it is
a 9. ax. plain that AD is *a* not equall to AC.

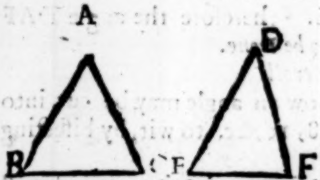
2. *Case.* If the point D be placed within the triangle ACB, then draw the line CD, and produce BDF, and BCE. Now you would have AD = AC, then the

the angle ADC *b* = ACD; as also, because BD *c* = BC, the angle FDC = *b* ACD. therefore is the angle FDC ⊏ *d* ACD. that is, the angle FDC ⊏ ADC. *d Which is imposs.* b 5. 1. c suppos. d 9. ax.

3. *Case.* If D falls without the triangle ACB, let CD be joyned.

Again, the angle ACD *e* = ADC, and the angle BCD *e* = BDC. *f* Therefore the angle ACD ⊏ BDC, *viz.* the angle ADC ⊏ BDC. *Which is impossible.* Therefore, &c. e 5. 1. f 9. ax.

PROP. VIII.



If two triangles ABC, DEF *have two sides* AB, AC *equall to two sides* DE, DF, *each to each; and the base* BC *equall to the base* EF, *then the angles conteined under the equall right lines shall be equall,* viz. A *to* D.

Because BC *a* = EF, if the base BC be laid on the base EF, *b* they will agree: therefore whereas AB *c* = DE, and AC = DF, the point A will fall on D (for it cannot fall on any other point, by the precedent proposition) and so the sides of the angles A and D are coincident; *d* wherefore those angles are equall. *Which was to be Dem.* a hyp. b ax. 8. c hyp. d 8. ax.

Coroll.

1. Hence, Triangles mutually equilateral are also mutually * equiangular. x 4. 1.

2. Triangles mutually equilateral * are equall one to the other.

PROP.

PROP. IX.

To bisect, or divide into two equall parts, a right-lined angle given B A C.

a 3. 1. a Take A D = to A E, and draw the line D E; upon

b 2. 1. which b make an equilateral triangle D F E. draw the right line A F; it shall bisect the angle.

c constr. For A D c = A E, and the side A F is common,

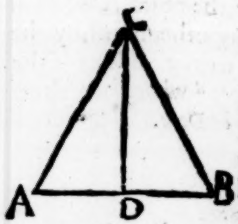
d 8. 1. & the base D F c = F E. d therefore the angle DAF = E A F. *Which was to be Done.*

Coroll.

Hence it appears how an angle may be cut into any equall parts, as 4, 8, 16, &c. to wit, by bisecting each part again.

The method of cutting angles into any equall parts required, by a Rule and Compasse, is as yet unknown to Geometricians.

PROP. X.



To bisect a right line given A B.

Upon the line given A B

a 1. 1. a erect an equilateral triangle

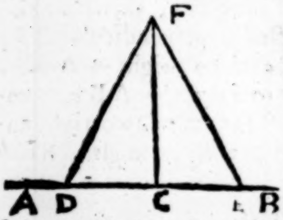
b 9. 1. A B C; and b bisect the angle C with the right line C D. That line shall also bisect the line given A B.

c constr. For A C c = B C, & the side C D is common, &

d 4. 1. the angle A C D c = B C D. therefore A D d = B D. *Which was to be Done.*

The practise of this and the precedent proposition is easily shewn by the construction of the 1. prop. of this book.

PROP. XI.



From a point C *in a right line given* AB *to erect a right line* CF *at right angles.*

a Take on either side of the point given CD = CE. upon the right line DE *b* erect an equilateral triangle. draw the line FC, and it will be the perpendicular required. a 3. 1. b 1. 1.

For the triangles DFC, EFC are mutually *c* equilateral; *d* therefore the angle DCF = ECF. *e* therefore FC is perpendicular. *Which was to be Done.* c constr. d 8. 1. e 10. def.

The practise of this and the following is easily performed by the help of a square.

PROP. XII.

Upon an infinite right line given AB, *from a point given that is not in it, to let fall a perpendicular right line* CG.

From the centre C *a* describe a circle cutting the right line given AB in the points E & F. Then *b* bisect EF in G, and draw the right line CG, which will be the perpendicular required. a 3. post. b 10. 1.

Let the lines CE, CF be drawn. The triangles EGC, FGC are mutually *c* equilaterall. *d* therefore the angles EGC, FGC are equall, and by *e* consequence right. *e* Wherefore GC is a perpendicular. *Which was to be Done.* c constr. d 8. 1. e 10 def.

PROP. XIII.

When a right line AB *standing upon a right line* CD *maketh angles* ABC, ABD; *it maketh either two right angles, or two angles equall to two right.*

If

If the angles A B C, A B D be equall, *a* then they
a def.10. make two right angles; if unequall, then from the
b 11.1. point B *b* let there be erected a perpendicular B E.
c 19.ax. Because the angle A B C *c* $=$ to a right $+$ A B E,
d 3.ax. and the angle A B D *d* $=$ to a right $-$ A B E, there-
e 2.ax. fore shall be A B C $+$ A B D *e* $=$ to two right angles $+$ A B E $-$ A B E $=$ 2 right angles. *Which was to be Dem.*

Corollaryes.

1. Hence, if one angle ABD be right, the other ABC is also right; if one acute, the other is obtuse, and so on the contrary.

2. If more right lines then one stand upon the same right line at the same point, the angles shall be equall to two right.

3. Two right lines cutting each other make angles equall to four right.

4. All the angles made about one point make 4. right; as appears by Coroll. 2.

PROP. XIV.

If to any right line A B, *and a point therein* B, *two right lines, not drawn from the same side, do make the angles* A B C, A B D *on each side equall to two right, the lines* C B, B D *shall make one strait line.*

a 13.1. If you deny it, let C B, B E make one right line;
b hyp. then shall be the angle A B C $+$ A B E *a* $=$ 2 right
c 9.ax. angles *b* $=$ A B C $+$ A B D. *Which is c Absurd.*

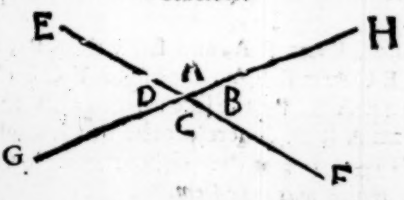
PROP. XV.

If two right lines A B, C D *cut through one another, then are the two angles which are opposite, viz.* C E B, A E D, *equall one to the other.*

For the angle A E C $+$ C E B
a 13.1. *a* $=$ to 2 right angles $=$ AEC $+$ AED; *b* therefore
b 3.ax. C E B $=$ A E D. *Which was to be Done.*

Schol.

Schol. 1.



If to any right line G H, and in it a point A, two right lines being drawn E A, A F, and not taken on the same side, make the verticall (or opposite) angles D and B equall, those right lines E A, A F, do meet directly and make one strait line.

For 2 right angles are *a* equall to the angle D + A *a* = B + A. *b* therefore E A, A F are in a strait line. *Which was to be Dem.* a 13. 1. b 14. 1.

Schol. 2.

If four right lines E A, E B, E C, E D, proceeding from one point E, make the angles vertically opposite equall the one to the other, each two lines, A E, E B, and C E, E D, are placed in one strait line.

For because the angle A E C + A E D + C E B + D E B *a* = to 4 right angles, therefore the angle A E C + A E D *b* = C E B + D E B = to two right angles. *c* Therefore C E D & A E B are strait lines. *Which was to be Dem.* a 4 cor. 13. 1. b hyp. & 2 ax. c 14. 1.

PROP. XVI.

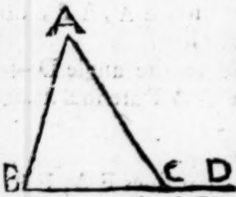
One side B C *of any triangle* A B C *being produc'd, the outward angle* A C D *will be greater then either of the inward and opposite angles* C A B, C B A.

Let the right lines A H, B E *a* bisect the sides A C, B C; *b* produce E F = B E, and H I a 10. 1. & 1. post. b 3. 1.

$b =$ A H. and join F C, and I C; and produce A C G.

c constr. d 15. 1. e 4. 1. Because C E $c =$ E A, and E F $c =$ E B, and the angle F E C $d =$ B E A, the angle E C F e shall be equall to E A B. By the like argument is the angle I C H $=$ A B H. Therefore the whole angle ACD f 15. 1. g 9. ax. (f B C G) g is greater then either the angle C A B or A B C. *Which was to be Dem.*

PROP. XVII.



Two angles of any triangle A B C, *which way soever they be taken, are lesse then two right angles.*

Let the side B C be produced. Because the angle ACD
a 13. 1. b 16. 1. c 4 ax. $+$ A C B $a =$ 2 right angles, and the angle ACD $b \sqsubset$ A, c therefore A $+$ ACB $\sqsupset$ then two right angles. After the same manner is the angle B$+$ACB $\sqsupset$ then two right. Lastly, the side A B being produced, the angle A $+$ B will be also lesse then two right angles. *Which was to be Dem.*

Coroll.

1. Hence it follows that in every triangle, wherein one angle is either right or obtuse, the two others are acute angles.

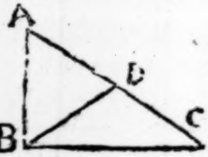
2. If a right line A E make unequall angles with another right line D, one acute A E D, the other obtuse A E C, a perpendicular A D let fall from any point A to the other line C D, shall fall on that side the acute angle is of.

For if A C, drawn on the side of the obtuse angle, be a perpendicular, then in the triangle AEC shall A E C $+$ A C E be greater then two right angles.
* 17. 1. * *Which is contrary to the precedent prop.*

3. All the angles of an equilateral triangle, and the 2 angles of an Isosceles triangle that are upon the base, are acute.

PROP.

PROP. XVIII.

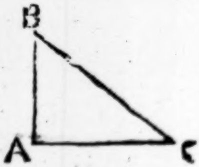


The greatest side A C *of every triangle* ABC *subtends the greatest angle* A B C.

From A C *a* take away A D = A B, and join BD. *b* Therefore is the angle ADB=ABD. But ADB *c* $\sqsubset$ C; therefore is ABD $\sqsubset$ C; *d* therefore the whole angle ABC $\sqsubset$ C. After the same manner, shall be ABC $\sqsubset$ A. *Which was to be Dem.*

a 3. 1.
b 5. 1.
c 16. 1.
d 9. ax.

PROP. XIX.

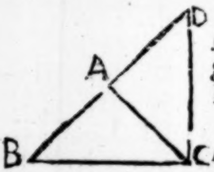


In every triangle ABC, *under the greatest angle* A *is subtended the greatest side* B C.

For if A B be supposed equall to B C, then will be the angle A *a* = C, which is contrary to the Hypothesis: and if AB $\sqsubset$ BC, then shall be the angle C *b* $\sqsubset$ A, which is against the Hypothesis. Wherefore rather B C $\sqsubset$ A B; and after the same manner BC $\sqsubset$ AC. *Which was to be Dem.*

a 5. 1.
b 18. 1.

PROP. XX.

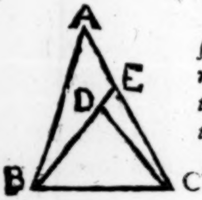


Of every triangle A B C *two sides* BA, AC, *any way taken, are greater then the side that remains* B C.

Produce the line BA, *a* & take AD=AC, & draw the line DC. *b* then shall the angle D be equall to ACD, *c* therefore is the whole angle BCD $\sqsubset$ D; *d* therefore BD (*e* BA + AC) $\sqsubset$ BC. *Which was to be Dem.*

a 3. 1.
b 5. 1.
c 9. ax.
d 19. 1
e constr. & 2. ax

PROP. XXI.

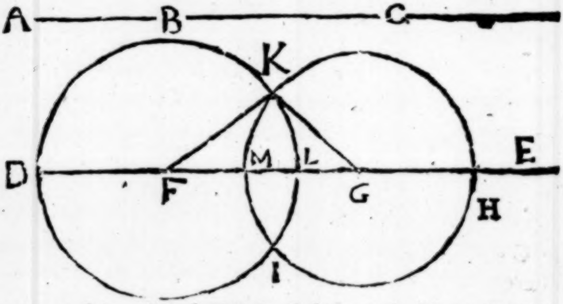


If from the utmost points of one side B C *of a triangle* A B C *two right lines* B D, C D *be drawn to any point within the triangle, then are both those two lines shorter then the two other sides of*

the triangle, BA, CA; *but do contain a greater angle,* BDC.

Let B D be produced to E. Then is C E + E D
a 20. 1. *a* $\sqsubset$ CD. adde BD common to both, *b* then shall be BD + DE + EC $\sqsubset$ CD + BD. Again, BA + AE
b 4. ax. *a* $\sqsubset$ BE. *b* therefore BA + AC $\sqsubset$ BE + E C. Wherefore 1. BA + AC $\sqsubset$ BD + DC. 2. The angle BDC
c 16. 1. *c* $\sqsubset$ DEC *c* $\sqsubset$ A. Therefore the angle BDC $\sqsubset$ A. *Which was to be Dem.*

PROP. XXII.



To make a triangle FKG *of three right lines* FK, FG, GK *which shall be equall to three right lines given* A, B, C. *Of which it is necessary that any two taken together be longer then the third.*

a 3. 1. From the infinite line DE *a* take DF, FG, GH equall to the lines given A, B, C. Then if from the
b 3. post. *b* centres F and G by the distances of F D and G H, two circles be drawn cutting each other in K, & the right lines KF, KG be joined, the triangle FKG
c 15. def. shall be made, *c* whose sides FK, FG, GK are equall
d 1. ax. to the three lines DF, FG, GH *d* that is to the three lines given A, B, C. *Which was to be Done.*

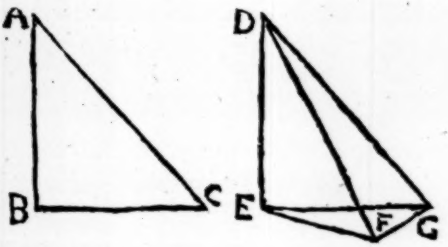
PROP. XXIII.

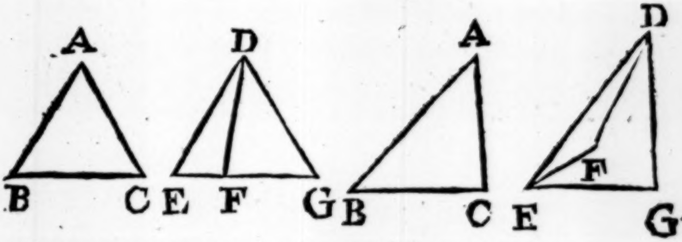
At a point A *in a right line given* A B, *to make a right-lined angle* A *equall to a right-lined angle given* D.

a 1. post. *a* Draw the right line

line C F cutting the sides of the angle given any wayes; *b* make AG = CD; upon AG *c* raise a trian- b 3. 1.
gle equilateral to the former C D F, so that A H be b 22. 1.
equall to D F, and GH to CF. then shall you have d 8. 1.
the angle A *d* = D. *Which was to be Done.*

PROP. XXIV.





If two triangles A B C , D E F *have two sides of the one triangle* AB, AC *equall to two sides of the other triangle* DE, DF, *each to other, and have the angle* A *greater then the angle* EDF *conteined under the equall right lines, they shall also have the base* B C *greater then the base* EF.

a Let the angle E D G be made equall to A, and a 23. 1.
the side DG *b* = DF *c* = AC ; & let EG, and FG b 3. 1.
be joyned. c *hyp.*

1. *Case.* If E G fall above EF; Because AB *d* = d *hyp*
D E, and A C *e* = DG, and the angle A *e* = EDG, e *constr.*
f therefore is BC = EG. But because DF *e* = DG, f 4. 1.
g therefore is the angle DFG = DGF; *h* therefore
is the angle DFG ⊏ EFG, and by consequence the g 5. 1.
angle EFG *h* ⊏ EGF. *k* wherefore EG (BC) ⊏ EF. h 9. *ax.* k 19. 1.

 2. *Case.*

2. *Case.* If the base EF falls in the same place with
l 9 ax. EG, [l] it is evident that EG (BC) $\sqsubset$ EF.

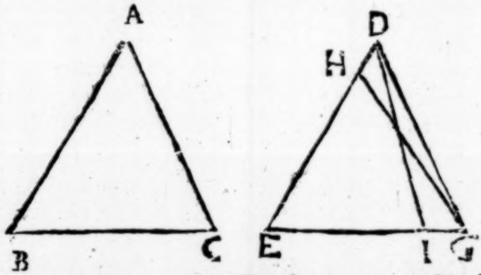
3. *Case.* If EG fall below EF, then because DG +
m 21. 1. GE [m] $\sqsubset$ DF + FE, if from both DG, DF be taken
n 5. ax. away, which are equall, EG (BC) remains [n] $\sqsubset$ EF. *Which was to be Dem.*

PROP. XXV.

If two triangles ABC, DEF *have two sides* AB, AC *equall to two sides* DE, DF, *each to other, and have the base* BC *greater then the base* EF, *they shall also have the angle* A *contained under the equall right lines greater then the angle* D.

a 4. 1. For if the angle A be said to be equall to D, [a] then is the base BC = EF, which is against the Hypothesis. If it be said the angle A $\sqsupset$ D, then [b] will
b 24. 1. be BC $\sqsupset$ EF, which is also against the Hypothesis. Therefore BC $\sqsubset$ EF. *Which was to be Dem.*

PROP. XXVI.



If two triangles BAC, EDG *have two angles of the one* B, C *equall to two angles of the other* E, DGE, *each to his correspondent angle, and have also one side of the one equall to one side of the other, either that side which lyeth betwixt the equall angles, or that which is subtended under one of the equall angles; the other sides al-*

so

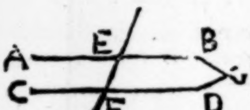
so of the one shall be equall to the other sides of the other, each to his correspondent side, and the other angle of the one shall be equall to the other angle of the other.

1. *Hypothesis.* Let BC be equall to EG, which are the sides that lye between the equall angles. Then I say BA = ED, and AC = DG, & the angle A = EDG. For if it be said that ED ⊏ BA, then a let EH a 3. 1.
be made equall to BA, & let the line GH be drawn.

Because AB b = HE, and BC c = EG, and the b suppos.
angle B c = E, therefore shall be the angle EGH c hyp.
d = C e = DGE. f *Which is absurd.* After the same d 4. 1.
manner let AC be equall to DG, d then will the angle e hyp.
A be equall to EDG. f 9 ax.

2. *Hyp.* Let AB be equall to ED. Then I say BC = EG, and AC = DG, and the angle A = EDG. For if EG be greater then BC, make EI = BC, and joyn the line DI. Now because AB g = ED, and g hyp.
BC h = EI, and the angle G g = E; therefore h suppos.
will be the angle EID k = C l = EGD. m *Which is* k 4. 1.
absurd. Therefore is BC = EG, and so as before l hyp.
AC = DG, and the angle A = EDG. *Which was to* m 16. 1.
be Dem.

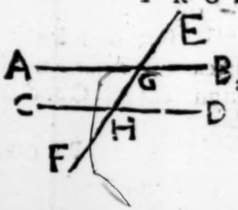
PROP. XXVII.



If a right line EF *falling upon two right lines* AB, CD *make the alternate angles* AEF, DFE, *equall the one to the other, then are the right lines* AB, CD *parallel.*

If AB, CD be said not to be parallel, produce them till they meet in G. which being supposed, the outward angle AEF will be a greater then the in- a 16. 1.
ward angle DFE, to which it was equall by Hypothesis. *Which is repugnant.*

PROP. XXVIII.



If a right line EF *falling upon two right lines* AB, CD *make the outward angle* AGE *of the one line equall to* CHG *the inward and opposite angle*

of

of the other on the same side, or make the inward angles on the same side AGH, CHG *equall to two right angles, then are the right lines* AB, CD *parallel.*

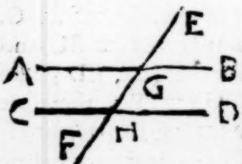
a 15. 1.
b 27. 1.

Hyp. 1. Because by Hypothesis the angle AGE = CHG, *a* therefore are BGH, CHG alternate angles and equall: And *b* therefore are AB and CD parallel.

a 13. 1.
b 3. ax.
c 27. 1.

Hyp. 2. Because by Hypothesis the angle AGH + CHG = to two right, *a* = AGH + BGH, *b* thence is CHG = BGH; and *c* therefore AB, CD are parallel. *Which was to be Dem.*

PROP. XXIX.

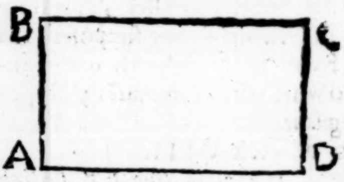


If a right line EF *fall upon two parallels,* AB, CD, *it will both make the alternate angles* DHG, AGH *equall each to other, and the outward angle* BGE *equall to the inward and opposite angle on the same side* DHE, *as also the inward angles on the same side* AGH, CHG *equall to two right angles.*

a 13 ax.
b 13. 1.
c 13. ax.
d 15. 1.

It is evident that AGH + CHG = 2 right angles; *a* otherwise A B, CD would not be parallel, which is contrary to the Hypothesis: But moreover the angle DHG + CHG *b* = 2 right; therefore is DHG *c* = AGH *d* = BGE. *Which was to be Dem.*

Coroll.



Hence it follows that every parallelogram A C having one angle right A, the rest are also right.

a 29. 1.
b 3. ax.

For A + B *a* = 2 right angles. Therefore, whereas A is right, *b* must B be also right. By the same argument are C and D right angles.

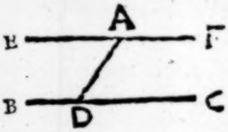
PROP.

PROP. XXX.

Right lines (AB, CD) parallel to one & the same right line EF, are also parallel the one to the other.

Let G I cut the three right lines given any wayes. then because AB, EF are parallel, will be the angle AGI *a* = EHI. also because CD a 19. 1.
and EF are parallel, will be the angle EHI *a* = b 1 ax.
DIG. *b* Therefore the angle AGI = DIG. *c* whence c 27. 1.
AB and CD are parallel. *Which was to be Dem.*

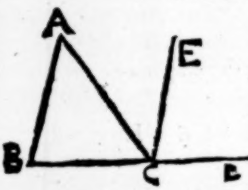
PROP. XXXI.



From a point given A to draw a right line AE parallel to a right line given BC.

From the point A draw a right line AD to any point of the given right line; with which at the point thereof
a A make an angle DAE = ADC. *b* then will AE a 23. 1.
and BC be parallel. *Which was to be Done.* b 27. 1.

PROP. XXXII.



Of any triangle ABC one side BC being drawn out, the outward angle ACD shall be equall to the two inward opposite angles A, B. and the three inward angles of the triangle, A, B, ACB, shall be equall to two right angles.

From C *a* draw CE parallel to BA. Then is the a 31. 1.
angle A *b* = ACE, and the angle B *b* = ECD. b 29. 1.
Therefore A + B *c* = ACE + ECD *d* = ACD. c 2. ax.
Which was to be Dem. d 19. ax.

I affirm ACD + ACB *e* = 2 right angles; e 13. 1.
f therefore A + B + ACB = 2 right angles. *Which* f 1. ax.
was to be Dem.

Coroll.

1. The three angles of any triangle taken together are

are equall to three angles of any other triangle taken together. From whence it follows,

2. That if in one triangle, two angles (taken severally, or together) be equall to two angles of another triangle (taken severally, or together) then is the remaining angle of the one equall to the remaining angle of the other. In like manner, if two triangles have one angle of the one equall to one of the other, then is the sum of the remaining angles of the one triangle equall to the summe of the remaining angles of the other.

3. If one angle in a triangle be right, the other two are equall to a right. Likewise, that angle in a triangle which is equall to the other two, is it self a right angle.

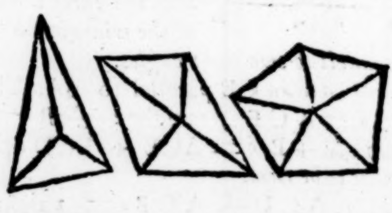
4. When in an Isosceles the angle made by the equall sides is right, the other two upon the base are each of them half a right angle.

5. An angle of an equilateral triangle makes two third parts of a right angle. For $\frac{1}{3}$ of two right angles is equall to $\frac{2}{3}$ of one.

Schol.

By the help of this proposition you may know how many right angles the inward and outward angles of a right-lined figure make; as may appear by these two following Theoremes.

THEOREME I.



All the angles of a right-lined figure do together make twice as many right angles, bating four, as there are sides of the figure.

From any point within the figure let right lines be

be drawn to all the angles of the figure, which shall resolve the figure into as many triangles as there are sides of the figure. Wherefore, whereas every triangle affords two right angles, all the triangles taken together will make up twice as many right angles as there are sides. But the angles about the said point within the figure make up four right; therefore, if from the angles of all the triangles you take away the angles which are about the said point, the remaining angles, which make up the angles of the figure, will make twice as many right angles, bating four, as there are sides of the figure. *Which was to be Dem.*

Coroll.

Hence, All right-lined figures of the same species have the sums of their angles equall.

THEOREME II.

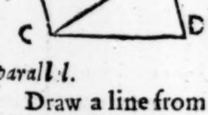
All the outward angles of any right-lined figure, taken together, make up four right angles.

For all the severall inward angles of a figure with the severall outward angles of the same make two right angles; therefore all the inward angles together with all the outward, make twice as many right angles as there are sides of the figure; But (as it was now shewn) all the inward angles with four right make twice as many right as there are sides of the figure; therefore the outward angles are equall to 4 right angles. *Which was to be Dem.*

Coroll.

All right-lined figures of whatsoever species have the summes of their outward angles equall.

PROP. XXXIII.

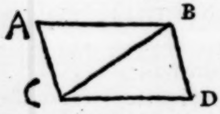


If two equall and parallel lines AB, CD *be joyned together with two other right lines* A C, B D, *then are those lines also equall & parallel.*

Draw a line from C to B. Now because AB and CD are parallel, and the angle ABC a = BCD; a 29. 1.
and also by hypothesis AB = CD, and the side CB com-

common, therefore is A C b = B D, and the angle
b 4. 1. ACB b = DBC. c whence also AC, BD are parallel.
c 27. 1.

PROP. XXXIV.



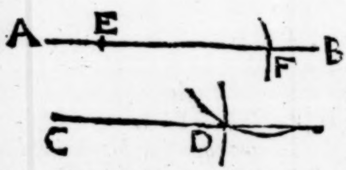
In parallelograms, as ABCD, *the opposite sides* AB, CD, *and* AC, BD, *are equall each to other; and the opposite angles* A, D, *and* ABD, AC *are also equall; and the diameter* BC *bisects the same.*

a hyp. Because AB, CD a are parallel, b therefore is the angle ABC = BCD. Also because AC, BD are a parallel, b therefore is the angle ACB = CBD;
b 29. 1.
c 2. ax. c therefore the whole angle ACD = ABD. After the same manner is A = D. Moreover because the angles ABC, ACB lye at each end of the side CB, and are equall to BCD, CBD, d therefore is AC =
d 26. 1. BD, and AB d = CD, and so the triangle ABC = CBD. *Which was to be Dem.*

Schol.

Every four-sided figure ABDC *having the opposite sides equall, is a parallelogram.*

For by 8. 1. the angle ABC = BCD; a wherefore
a 27. 1. AB, CD are parallel. In like manner is the angle BCA = CBD; a wherefore AC, BD are also parallel.
b 35. def. 1. b Therefore ABCD is a parallelogram. *Which was to be Dem.*

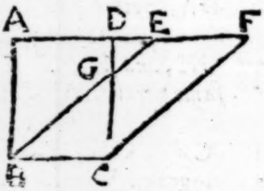


From hence may be learned how to draw a parallel CD to a right line given AB, through a point assigned C.

Take in the line AB any point, as E. From the centres E and C at any distance draw two equall circles EF, CD. From the centre F by the space of EC draw a circle FD, which shall cut the former circle CD in the point D. Then shall the line drawn CD be parallel to AB. for as it was before demonstrated, CEFD is a parallelogram.

PROP.

PROP. XXXV.



Parallelograms, BCDA, BCFE, *which stand upon the same base* BC, *and between the same parallels* AF, BC, *are equall one to the other.*

For AD a = BC a = EF. adde DE common to both, b then is AE = DF. But also AB a = DC, and the angle A c = CDF. d Therefore is the triangle ABE = DCF. take away DGE common to both triangles, e then is the Trapezium ABGD = EGCF. adde BGC common to both, f then is the parallelogram ABCD = EBCF. *Which was to be Dem.*

a 34. 1.
b 2 ax.
c 29. 1.
d 4 1.
e 3 ax.
f 2. ax.

The demonstration of any other cases is not unlike, but much more plain and easy.

Schol.

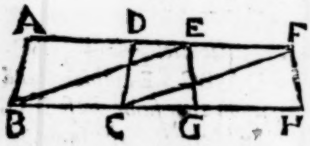
If the side AB of a right-angled parallelogram ABCD be conceived to be carried along perpendicularly through the whole line BC, or BC through the whole line AB, the Area or conteut of the Rectangle ABCD shall be produc'd by that motion. Hence a rectangle is said to be made by the drawing or multiplication of two contiguous sides. For examples sake; let AB be supposed four foot, and BC three: draw 3 into 4, there will be produced 12 square feet for the Area of the Rectangle.

This being supposed, the dimension of any parallelogram (*EBCF) is found out by this theoreme. For the Area thereof is produced from the altitude BA drawn into the base BC. So the Area of the parallelogram AC = EBCF, is made by the drawing of BA into BC, therefore, &c.

* See the fig. of prop. 35.

PROP.

PROP. XXXVI.

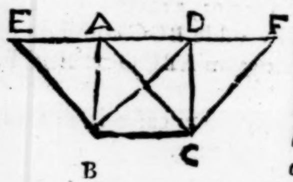


Parallelograms, BCDA, GHFE, *standing upon equall bases* BC, GH, *and betwixt the same parallels* AF, BH, *are equall one to the other.*

a hyp.
b 34. 1.
c 33. 1.
d 35. 1.

Draw BE, & CF. Because BC *a* = GH *b* = EF, *c* therefore is BCFE a parallelogram. Whence the parallelogram BCDA *d* = BCFE *d* = GHFE. *Which was to be Dem.*

PROP. XXXVII.



Triangles, BCA, BCD, *standing upon the same base* BC, *and between the same parallels* BC, EF, *are equall one to the other.*

a 31. 1.
b 34. 1.
c 35. 1. and 7. ax.

a Draw BE parallel to CA, *a* and CF parallel to BD. Then is the triangle BCA *b* = $\frac{1}{2}$ of the parallelogram BCAE *c* = $\frac{1}{2}$ BDFC *b* = BCD. *Which was to be Dem.*

PROP. XXXVIII.

Triangles, BCA, EFD, *set upon equall bases* BC, EF, *and between the same parallels* GH, BF, *are equall the one to the other.*

a 34. 1.
b 36. 1. and 7. ax.
c 34. 1.

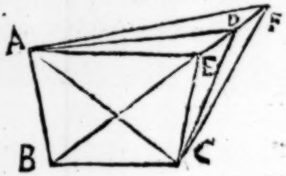
Draw BG parallel to CA, & FH parallel to ED. Then is the triangle BCA *a* = $\frac{1}{2}$ Pgr. BCAG *b* = $\frac{1}{2}$ EDHF *c* = EFD. *Which was to be Dem.*

Schol.

If the base BC be greater then EF, then is the triangle BAC $\sqsubset$ EDF, and so on the contrary.

PROP.

PROP. XXXIX.



Equall triangles BCA, BCD, *standing on the same base* BC, *and on the same side, they are also between the same parallels* AD, BC.

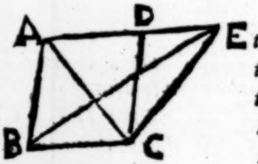
If you deny it, let another line AF be parallel to BC; and let CF be drawn. Then is the triangle CBF a 37. 1.
a = CBA *b* = CBD. *c Which is absurd.* b hyp. c 9 ax.

PROP. XL.

Equall triangles BCA, EFD, *standing upon equall bases* BC, EF, *and on the same side, they are betwixt the same parallels.*

If you deny it, let another line AH be parallel to BF, and let FH be drawn. Then is the triangle EFH *a* = BCA *b* = a 38. 1.
EFD. *c Which is absurd.* b hyp. c 9. ax.

PROP. XLI.



If a Pgr. ABCD *have the same base* BC *with the triangle* BCE, *and be between the same parallels* AE, BC, *then is the Pgr.* ABCD *double to the triangle* BCE.

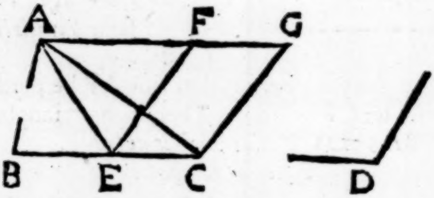
Let the line AC be drawn. Then is the triangle BCA *a* = BCE. therefore is the Pgr. ABCD *b* = a 37. 1.
2 BCA *c* = 2 BCE. *Which was to be Dem.* b 34. 1. c 6. ax.

Schol.

From hence may the Area of any triangle BCE be found. for whereas the Area of the Pgr. ABCD is produced by the altitude drawn into the base, therefore shall the Area of a triangle be produced by half the altitude drawn into the base, or half the base drawn

drawn into the altitude. as if so be the base BC be 8, and the altitude 7, then is the area of the triangle BCE 28.

PROP. XLII.

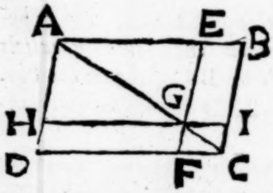


To make a Pgr. ECGF *equall to a triangle given* ABC *in an angle equal to a right-lined angle given* D.

a 31. 1. Through A *a* draw A G parallel to B C. *b* make
b 23. 1. the angle B C G = D. *c* bisect the base B C in E,
c 10. 1. and draw E F parallel to C G. then is the probleme resolved.

For AE being drawn, the angle ECG is equall to
d 38. 1. D by construction, and the triangle BAC *d* = 2
e 41. 1. AEC *e* = Pgr. ECGF. *Which was to be Done.*

PROP. XLIII.

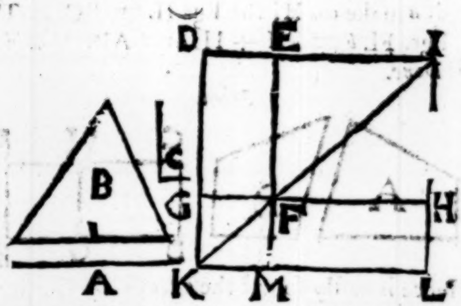


In every Pgr. ABCD, *the complements* DG, GB *of those Pgrs* HE, FI, *which stand about the diameter, are equall one to the other.*

For the triangle ACD
a 34 1. *a* = ACB, and the triangle AGH *a* = AGE, and
b 3. ax. the triangle GCF *a* = GCI. *b* Therefore the Pgr. DG = BG. *Which was to be Dem.*

PROP.

PROP. XLIV.



Upon a right line given A, *to make a Parallelogram* FL *at a right-lined angle given* C, *equall to a triangle given* B.

a Make a Pgr. FD = to the triangle B, so that the angle GFE may be equall to C. Produce GF till FH be equall to the line given A. through H *b* draw IL parallel to EF, which let DE produced meet in the point I. let DG drawn forth meet with a right line drawn from I in the point K. through K *b* draw KL parallel to GH, with which let EF drawn out meet at M, and IH at L. Then shall FL be the Pgr. required. a 42. 1. b 31. 1.

For the Pgr. FL *c* = FD = B, *d* and the angle MFH = GFE = C. *Which was to be Done.* c 43. 1. d 15. 1.

PROP. XLV.

Upon a right line given FG *to make a Pgr.* FL *equall to a right-lined figure given* ABCD, *at a right-lined angle given* E.

Resolve the right-lined figure given into triangles
a 44.1. BAD, BCD. then *a* make a Pgr. FH = BAD, so that the angle F may be equall to E. FI being produced, *a* make on HI the Pgr. IL = BCD. Then is
b 19.ax. the Pgr. FL *b* = FH + IL *c* = ABCD. *Which was*
c constr. *to be Done.*

Schol.

Hence is easily found the excesse, HE, whereby any right-lined figure, A, exceeds a lesse right-lined figure, B; namely, if to some right line, CD, both be applyed, Pgr. DF = A, and DH = B.

PROP. XLVI.

Upon a right line given AD *to describe a square* AC.

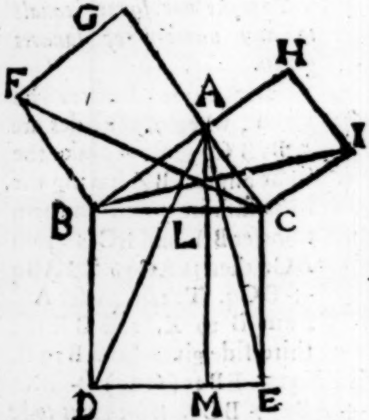
a 11. 1. *a* Erect two perpendiculars AB, DC, *b* e-
b 3. 1. quall to the line given AD; then joyn BC, & the thing required is done.

c constr. For, whereas the angle A+D *c* = 2 right, *d* there-
d 28.1. fore are AB, DC parallel. But they are also *e* equall;
e constr. *f* therefore AD, BC are both parallel and equall;
f 34. 1. therefore the figure AC is a Pgr. and equilateral.
g sch.29.1. Moreover the angles are all right, *g* because one,
h 29.def. A, is right; *h* therefore AC is a square. *Which was to be Done.*

After the same manner you may easily describe a rectangle conteined under two right lines given.

PROP.

PROP. XLVII.



In right angled triangles, BAC, the square BE, which is made of the side BC that subtends the right angle BAC, is equall to both the squares BG, CH, which are made of the sides AB, AC conteining the right angle.

Join AE, and AD; and draw AM parallel to CE.

Because the angle DBC a = FBA, adde the angle ABC common to them both; then is the angle ABD = FBC. Moreover AB b = FB, and BD b = BC; c therefore is the triangle ABD = FBC. But the Pgr. BM d = 2 ABD, and the Pgr. d BG = 2 FBC (for GAC is one right line by Hypothesis, and 14.1.) e therefore is the Pgr. BM = BG. By the same way of argument is the Pgr. CM = CH. Therefore is the whole BE = BG + CH. *Which was to be Dem.*

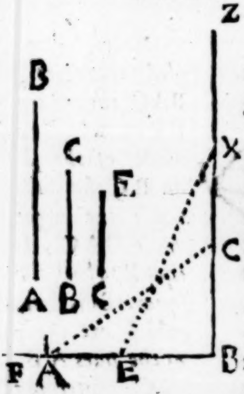
a 12. ax.
b 29. def.
c 4. 1.
d 41. 1.
e 6. ax.

Schol.

This most excellent and usefull theoreme hath deserved the title of *Pythagoras* his theoreme, because he was the inventer of it. By the help of which the addition and substraction of squares are performed; to which purpose serve the 2 following problemes.

PROBLEME I.

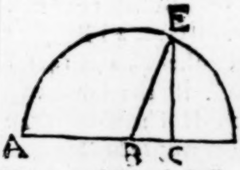
Andr. Tacq.



To make one square equall to any number of squares given.

Let three squares be given, whereof the sides are AB, BC, CE. a 11. 1. [a] Make the right angle FBZ having the sides infinite; and on them transfer BA and BC; join AC. b 47. 1. then is ACq [b] = ABq + BCq. Then transfer AC from B to X, and CE the third side given from B to E; join EX. [b] Then is EXq = EBq (CEq) + BXq c 2. ax. (ACq) [c] = CEq + ABq + BCq. *Which was to be Done.*

PROBLEME II.

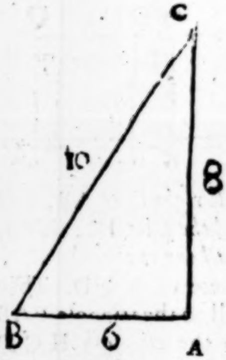


Two unequall right lines being given AB, BC, *to make a square equall to the difference of the two squares of the given lines,* AB, BC.

From the centre B, at the distance of BA, describe a circle; and from the point C erect a perpendicular CE meeting with the circumference in E; and draw BE. a 47. 1. b 3. ax. [a] Then is BEq (BAq) = BCq + CEq. [b] Therefore BAq − BCq = CEq. *Which was to be Done.*

PRO-

PROBLEME III.

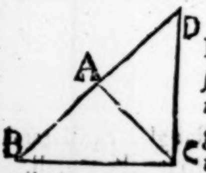


Any two sides of a right-angled triangle A B C *being known, to find out the third.*

Let the sides A B, A C encompassing the right angle, be, the one 6 foot, the other 8. Therefore, whereas ACq + ABq = 64 + 36 = 100 = BCq, thence is BC = √ 100 = 10. 47. 1.

Otherwise, let the sides AB, B C be known, the one 6 foot, the other 10. Therefore since BCq − ABq = 100 − 36 = 64 = ACq, 47. 1. thence is AC = √ 64 = 8. *Which was to be Done.*

PROP. XLVIII.



If the square made upon one side BC *of a triangle be equall to the squares made on the other sides of the triangle,* AB, AC, *then the angle* BAC *comprehended under those two other sides of the triangle* AB, AC, *is a right angle.*

Draw to the point A in AC a perpendicular line DA = AB, and join CD.

Now is a CDq = ADq + ACq = ABq + ACq = BCq. * Therefore is CD = BC. And therefore the triangles CAB, CAD are mutually equilaterall. Wherefore the angle CAB b = CAD c = a right angle. *Which was to be Dem.*

a 47. 1.
* See the following Theor.
b 8. 1.
c constr.

Schol.

We assumed in the demonstration of the last proposition, CD = BC, because CDq was equall to BCq: our assumption we prove by the following theoreme.

THEOREME.

The squares AF, CG *of equall right lines* AB, CD *are equall one to the other* : *And the sides* IK , LM *of equall squares* NK, PM, *are equall one to the other.*

1. *Hypothesis.* Draw the diameters EB, HD. Then it is evident that AF is *a* equall to the triangle EAB twice taken , and *b* equall to the triangle HCD twice taken, and equall to *a* CG. *Which was to be Dem.*

a 34. 1.
b 4. 1. & 6. ax.

2. *Hyp.* If it may be, let LM be greater then IK. Make LT = IK, and let LS be *a* equall to LTq. Therefore is LS *b* = NK *c* = LQ. *d Which is Absurd.*

a 46 1.
b 1. part.
c hyp.
d 9. ax.

Coroll.

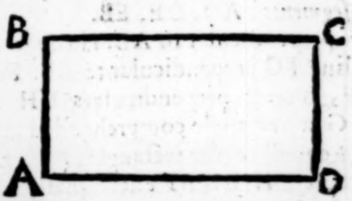
After the same manner any rectangles equilateral one to another, are demonstrated to be also equall.

The End of the first Book.

THE

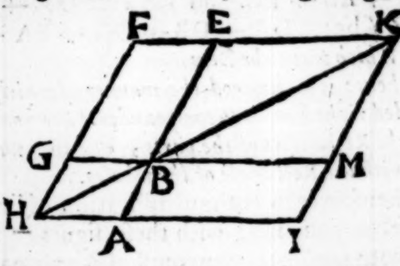
THE SECOND BOOK OF EUCLIDE'S ELEMENTS.

Definitions.



I. EVery right-angled Parallelogram ABCD is said to be conteined under two right lines AB, AD comprehending a right angle.

Therefore when you meet with such as these, the rectangle under BA, AD, *or for shortnesse sake the rectangle* BAD, *or* BA x AD (or ZA, for ZxA) *the rectangle meant is that which is conteined under the right lines* BA, AD *set at right angles.*



II. In every Pgr. FHIK, any one of those parallelograms which are about the diameter together with its two complements is called a Gnomon. *As the Pgr.* FB + BI + GA (EHM) *is a Gnomon; and likewise the Pgr.* FB+BI+ EM(GKA) *is a Gnomon.*

PROP. I.

If two right lines AF, AB, *be given, and one of them* AB *divided into as many parts or segments as you please; the rectangle comprehended under the two whole right lines* AB, AF, *shall be equall to all the rectangles conteined under the whole line* AF *and the severall segments,* AD, DE, EB.

a 11. 1. a Set AF perpendicular to AB. Through F a draw an infinite line FG perpendicular to AF. From the points D, E, B erect perpendiculars DH, EI, BG. Then is AG a rectangle comprehended under AF,

b 19. ax. 1. c 34. 1. AB, and is b equall to the rectangles AH, DI, EG, that is (because DH, EI, AF c are equall) to the rectangles under AF, AD, under AF, DE, under AF, EB. *Which was to be Dem.*

Schol.

If two right lines given be both divided into how many parts soever, the product of the whole multiplied into it self shall be the same with that of the parts multiplied into themselves.

For let Z be $= A + B + C$, and $Y = D + E$;

a 1. 2. then, because DZ a $= DA + DB + DC$, and EZ a $= EA + EB + EC$, and YZ a $= DZ + EZ$,

b 2. ax. b shall ZY be $= DA + DB + DC + EA + EB + EC$. *Which was to be Dem.*

From hence is understood the manner of multiplying compounded right lines into compounded. For you must take all the Rectangles of the parts, and they will present you with the Rectangle of the wholes.

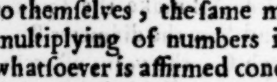
But whensoever in the multiplication of lines into themselves you meet with these signes − intermingled with these +, you must also have particular regard to the signes. For of + multiply'd into − ariseth −; but of − into − ariseth +. *ex. gr.* let +A be to be multiply'd into B − C; then because + A is not affirmed of all B, but only of a part of it, whereby it exceeds C, therefore AC must remain de-

denied; so that the product will be $AB - AC$. Or thus; because B consists of the parts C and $B - C$, * thence $AB = AC + A \times B - C$. take away AC from either, then $AB - AC = A \times B - C$. In like manner if $- A$ be to be multiply'd into $B - C$, then seeing by reason of the figure $-$, that A is not denied of all B, but only of so much as it exceeds C, therefore AC must remain affirmed. whence the product will be $- AB + AC$. Or thus; because $AB = AC + A \times B - C$; take away all throughout, and there will be $- AB = AC - A - \times B + C$; adde AC to either, and there will be $- AB + AC = A \times B - C$. * 1.1.

This being sufficiently understood, the nine following propositions, and innumerable others of that kind, arising from the comparing of lines multiply'd into themselves (which you may find done to your hand in *Vieta* and other Analytical writers) are demonstrated with great facility, by reducing the matter for the most part to almost a simple work.

Furthermore, * it appears that the product of any magnitude multiply'd into the parts of any number is equall to the product of the same multiply'd into the whole number: As $5\,A + 7\,A = 12\,A$, and $4\,A \times 5\,A + 4\,A \times 7\,A = 4\,A \times 12\,A$. Wherefore what is here deliver'd of the multiplying of right lines into themselves, the same may be understood of the multiplying of numbers into themselves. so that whatsoever is affirmed concerning lines in the nine following theoremes, holds good also in numbers; seeing they all immediately depend and are deriv'd from this first. * 19. ax.

PROP. II.

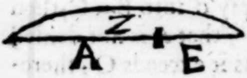


If a right line Z be divided any-wise into two parts, the rectangles comprehended under the whole line Z and each of the segments A, E, are equall to the square made of the whole line Z.

I say that $ZA + ZE = Zq$. For take $B = Z$; a then is $BA + BE = BZ$. that is (because $B = Z$) $ZA + ZE = Zq$. *Which was to be Dem.*

a 1. 2.

PROP. III.

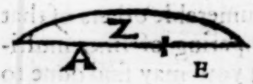


If a right line Z be divided any-wise into two parts, the rectangle comprehended under the whole line Z and one of the segments E is equall to the rectangle made of the segments A, E, and the square described on the said segment E.

a 1. 2.

I say $ZE = AE + Eq$. a For $EZ = EA + Eq$.

PROP. IV.

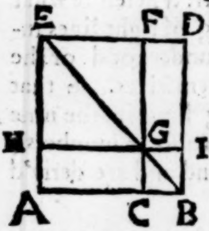


If a right line Z be cut any-wise into two parts, the square made of the whole line Z is equall both to the squares made of the segments A, E, and to twice a rectangle made of the parts A, E.

a 3. 2.
b 2. 2.
c 1. ax.

I say that $Zq = Aq + Eq + 2\ AE$. For ZA a $= Aq + AE$, and ZE a $= Eq + EA$. Therefore whereas $ZA + ZE$ b $= Zq$, c thence is $Zq = Aq + Eq + 2\ AE$. *Which was to be Dem.*

PROP. V.



Otherwise thus; Upon the right line AB make the square AD, and draw the diameter EB; through C, the point wherein the line AB is divided, draw the perpendicular CF; and through the point G draw HI parallel to AB.

d 4 cor. 32. 1.
e 32. 1.
f 6. 1.
g 34 1.
h 29. def.
k 19. ax. 1.

Because the angle $EHG = A$ is a right angle, and AEB is d half a right, e therefore is the remaining angle HGE half a right angle. Therefore is HE f $=$ HG g $=$ EF g $=$ AC, so that HF h is the square of the right line A C. After the same manner is CI proved to be CBq. Therefore AG, GD are rectangles under AC, CB. wherefore the whole square AD k $= ACq + CBq + 2\ ACB$. *Which was to be Dem.*

Co-

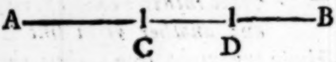
Coroll.

1. Hence it appears that the Pgrs which are about the diameter of a square are also squares themselves.

2. That the diameter of any square bisects it's angles.

3. That if $A = \frac{1}{2} Z$, then is $Zq = 4\,Aq$, and $Aq = \frac{1}{4} Zq$. As on the contrary, if it be so that $Zq = 4\,Aq$, then is $A = \frac{1}{2} Z$.

PROP. V.



If a right line AB *be cut into equall parts* AC, CB, *and into unequall parts* AD, DB, *the rectangle comprehended under the unequall parts* AD, DB, *together with the square that is made of the difference of the parts* CD, *is equall to the square that is made of the half line* CB.

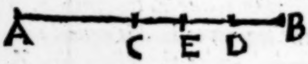
I say that $CBq = ADB + CDq$.

For these are all equall; $\begin{cases} CBq. \\ {}^{a}\,CDq + CDB + DBq + CDB. \\ CDq + {}^{b}\,CBD\,({}^{c}AC \times BD) + CDB. \\ CDq + {}^{d}\,ADB. \end{cases}$ a 4.2. b 3.2. c hyp. d 1.2.

This theoreme is somewhat differently express'd and more easily demonstrated thus; *A Rectangle made of the summe and the difference of two right lines* A, E, *is equall to the difference out of them.*

For if $A + E$ be multiply'd into $A - E$, * there ariseth $Aq - AE + EA - Eq = Aq - Eq$. *Which was to be Dem.*

Schol.



If the line AB be divided otherwise, (*viz.*) nearer to the point of bisection, in E; Then is AEB ⊏ ADB.

For AEB ${}^{a}=$ CBq − CEq. and ADB ${}^{a}=$ CBq − CDq. Therefore, whereas CDq ⊏ CEq, thence is AEB ⊏ ADB. *W. W. to be Dem.* a 5.2. & 3.ax.

Co-

Coroll.

1. Hence is $ADq + DBq \sqsubset AEq + EBq$. For $ADq + DBq + 2\,ADB$ ${}^{b}= ABq$ ${}^{b}= AEq + EBq + 2\,AEB$. Therefore because $2\,AEB \sqsubset 2\,ADB$, thence is $ADq + DBq \sqsubset AEq + EBq$. *W.W. to be Dem.*

b 4. 2.

c 3. ax.

2. Hence is $ADq + DBq - AEq$ ${}^{c}+ EBq = 2\,AEB - 2\,ADB$.

PROP. VI.

If a right line A *be divided into two equall parts, and another right line* E *added to the same directly in one right line, then the rectangle comprehended under the whole and the line added,* (viz. $A+E$,) *and the line added* E, *together with the square which is made of* $\frac{1}{2}$ *the line* A, *is equall to the square of* $\frac{1}{2}\,A+E$ *taken as one line.*

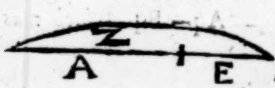
a 4 & 3. Cor. 4. 2.

I say that $\frac{1}{4}\,Aq$ (${}^{a}\,Q.\frac{1}{2}A$) $+ AE + Eq = Q.\frac{1}{2}A+E$. ^{a}For, $Q\frac{1}{2}A+E = \frac{1}{4}Aq + Eq + AE$. *Which was to be Dem.*

Coroll.

Hence it follows that if 3 right lines E, $E+\frac{1}{2}A$, $E+A$ be in arithmeticall proportion, then the rectangle conteined under the extreme termes E, $E+A$, together with the square of the difference $\frac{1}{2}A$, is equall to the square of the middle term $E+\frac{1}{2}A$.

PROP. VII.



If a right line Z *be divided any-wise into two parts, the square of the whole line* Z *together with the square made of one of the segments* E, *is equall to a double rectangle comprehended under the whole line* Z *and the said segment* E, *together with the square made of the other segment* A.

a 4. 2.
b 3. 2.

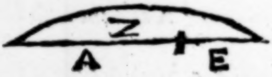
I say that $Zq + Eq = 2\,ZE + Aq$. For Zq ${}^{a}= Aq + Eq + 2\,AE$. and $2\,ZE$ ${}^{b}= 2\,Eq + 2\,AE$. *W. W. to be Dem.*

Coroll.

Hence it follows that the ſquare of the difference of any two lines Z, E, is equall to the ſquares of both the lines leſſe by a double rectangle comprehended under the ſaid lines.

c For $Zq + Eq - 2\,ZE = Aq = Q.\,Z - E$. c 7.2. and 3.ax.

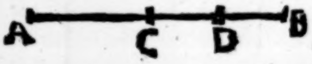
PROP. VIII.



If a right line Z *be divided any-wiſe into two parts, the rectangle comprehended under the whole line* Z *and one of the ſegments* E *four times, together with the ſquare of the other ſegment* A, *is equall to the ſquare of the whole line* Z *and the ſegment* E *taken as one line* $Z + E$.

I ſay that $4\,ZE + Aq = Q.\,Z + E$. For $2\,ZE$ *a* $= Zq + Eq - Aq$. Therefore $4\,ZE + Aq = Zq + Eq + 2\,ZE$ *b* $= Q.\,Z + E$. *W. W. to be Dem.* a 7.2. and 3.ax. b 4 2.

PROP. IX.



If a right line AB *be divided into equall parts* AC, CB, *and into unequall parts* AD, DB, *then are the ſquares of the unequall parts* AD, DB *together, double to the ſquare of the half line* AC, *and to the ſquare of the difference* CD.

I ſay that $ADq + DBq = 2\,ACq + 2\,CDq$. For $ADq + DBq$ *a* $= ACq + CDq + 2\,ACD + DBq$. a 4 2.
But $2\,ACD$ (*b* $2\,BCD$) $+ DBq$ *c* $= CBq\ (ACq) + CDq$. *d* Therefore $ADq + DBq = 2\,ACq + 2\,CDq$. *W. W. to be Dem.* b hyp. c 7.2. d 2.ax.

This may be otherwiſe deliver'd and more eaſily demonſtrated thus; *The aggregate of the ſquares made of the ſumme and the difference of two right lines* A, E, *is equall to the double of the ſquares made from thoſe lines.*

For $Q{:}\,A + E$ *a* $= Aq + Eq + 2\,AE$. and $Q{:}\,A - E$ *b* $= Aq + Eq - 2\,AE$. Theſe added together make $2\,Aq + 2\,Eq$. *Which was to be Dem.* a 4.2. b sch.7 2.

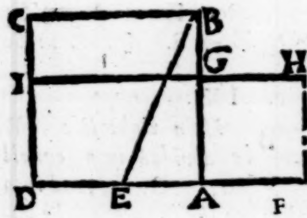
PROP.

PROP. X.

If a right line A *be divided into two equall parts, and another line be added in a right line with the same, then is the square of the whole line together with the added line (as being one line) together with the square of the added line* E, *double to the square of* $\frac{1}{2}$ A *and the added line* E, *taken as one line.*

I say that Eq + Q. A + E, *i. e.* [a] Aq + 2 Eq + 2
a 4. 2. AE = 2 Q. $\frac{1}{2}$ A + 2 Q $\frac{1}{2}$ A + E. For 2 Q. $\frac{1}{2}$ A [b] =
b cor 4. 2. $\frac{1}{2}$ Aq. And 2 Q. $\frac{1}{2}$ A + E [c] = $\frac{1}{2}$ Aq + 2 Eq + 2
c 4. 2. AE. *W. W. to be Dem.*

PROP. XI.



To cut a right line given AB *in a point* G, *so that the rectangle comprehended under the whole line* A B *and one of the segments* BG *shall be equall to the square that is made of the other segment* AG.

a 46 1. b 10. 1. Upon AB [a] describe the square AC. [b] Bisect the side AD in E, & draw the line EB; from the line EA produced take EF = EB. On AF make the square AH. Then is AH = ABxBG.

c 6. 2 d constr. e 47 1. f 3 ax. For HG being drawn out to I; the rectangle DH + EAq [c] = EFq [d] = EBq [e] = BAq + EAq: Therefore is DH [f] = BAq = to the square AC. Take away AI common to both, then remains the square AH = GC. that is, AGq = ABxBG. *W.W. to be Done.*

Schol.

a 6. 13. This proposition cannot be performed by numbers;* for there is no number that can be so divided, that the product of the whole into one part shall be equall to the square of the other part.

PROP.

PROP. XII.

In obtuse-angled triangles ABC, *the square that is made of the side* AC *subtending the obtuse angle* ABC *is greater then the squares of the sides* BC, AB *that contein the obtuse angle* ABC, *by* [a] *a double rectangle conteined under one of the sides* BC, *which are about the obtuse angle* ABC, *on which side produced the perpendicular* AD *falls, and under the line* BD, *taken without the triangle from the point on which the perpendicular* AD *falls to the obtuse angle* ABC.

I say that ACq = CBq + ABq + 2 CBxBD.

For these are all equall
- ACq.
- [a] CDq + ADq. a 47. 1.
- [b] CBq + 2 CBD + BDq + ADq. b 4 2
- CBq + 2 CBD + [c] ABq. c 47. 1.

Scholium.

Hence, *the sides of any obtuse-angled triangle* ABC *being known, the segment* BD *intercepted betwixt the perpendicular* AD *and the obtuse angle* ABC, *as also the perpendicular it self* AD *shall be easily found out.*

Thus. Let AC be 10, AB 7, CB 5. Then is ACq 100, ABq 49, CBq 25. And ABq + CBq = 74. Take that out of 100, then will 26 remain for 2 CBD. Wherefore CBD shall be 13; divide this by CB 5, there will $2\frac{3}{5}$ be found for BD. Whence AD will be found out by the 47. 1.

PROP. XIII.

In acute-angled triangles ABC, *the square made of the side* AB *subtending the acute angle* ACB, *is lesse then the squares made of the sides* AC, CB *comprehending the acute angle* ACB *by a double rectangle conteined under one of the sides* BC, *which are about the acute angle* ACB, *on which the perpendicular* AD *falls, and under the line* DC *taken within the triangle from the perpendicular* AD *to the acute angle* ACB.

I say that $ACq + BCq = ABq + 2\,BCD$.

a 47. 1.
b 7. 2.
c 47. 1.

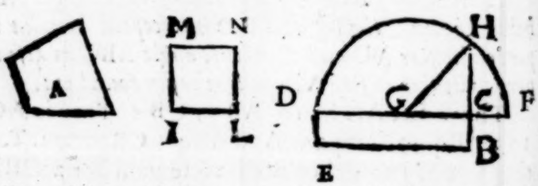
For these are equall $\begin{cases} ACq + BCq. \\ {}^{a}\ ADq + DCq + BCq. \\ {}^{b}\ ADq + BDq + 2\,BCD. \\ {}^{c}\ ABq + 2\,BCD. \end{cases}$

Coroll.

Hence, *The sides of an acute-angled triangle* ABC *being known, you may find out the segment* DC *intercepted betwixt the perpendicular* AD *and the acute angle* ABC, *as also the perpendicular it self* AD.

Let AB be 13, AC 15, BC 14. Take ABq (169) from ACq+BCq, that is, from $225 + 196 = 421$. Then remains 252 for 2 BCD. wherefore BCD will be 126. divide this by BC 14, then will 9 be found out for DC. From whence it follows $AD = \sqrt{:}\ 225 - 81 = 12$.

Prop. XIV.



To find a square ML *equall to a right-lined figure given* A.

a 45. 1.
b 10. 1.

^{a} Make the rectangle $DB = A$, and produce the greater side thereof DC to F, so that $CF = CB$. ^{b} bisect DF in G, about which as the centre at the distance of GF describe the circle FHD, and draw out CB till it touch the circumference in H. Then shall be $CHq = {}^{*}ML = A$.

* 46. 1.
c constr.
d 5. 2. and 3. ax.
e 47. 1. and 3. ax.

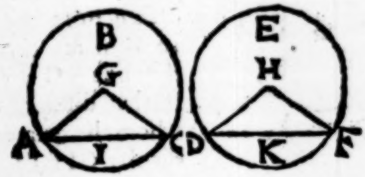
For let G H be drawn. Then is $A\ {}^{c} = DB\ {}^{c} = DCF\ {}^{d} = GFq - GCq\ {}^{e} = HCq\ {}^{*} = ML$. *W. W. to be Done.*

The End of the second Book.

LIB.

THE THIRD BOOK OF EUCLIDE'S ELEMENTS.

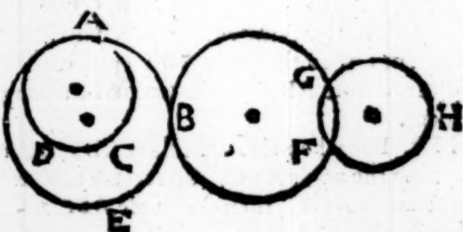
Definitions.



I. EQuall circles (GABC, HDEF) are such whose diameters are equall; or, from whose centres right lines drawn GA, HD, are equall.

II. A right line AB is said to touch a circle FED, when touching the same, and being produced, it cutteth it not.

The right line FG cuts the circle FED.

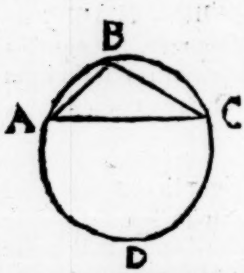


III. Circles DAC, ABE (and also FBG, ABE)

are said to touch one to the other, which touch, but cut not one the other.

The circle BFG cuts the circle FGH.

IV. In a circle GABD, right lines FE, KL are said to be equally distant from the centre, when perpendiculars GH, GN drawn from the centre G to them are equall. And that line BC is said to be furthest distant from it, on whom the greater perpendicular GI falls.



V. A segment of a circle (ABC) is a figure conteined under a right line AC, and a portion of the circumference of a circle ABC.

VI. An angle of a segment CAB, is that angle which is conteined under a right line C A and an arch of a circle AB.

VII. An angle ABC is said to be in a segment ABC, when in the circumference thereof some point B is taken, and from it right lines AB, CB, drawn to the ends of the right line A C, which is the base of the segment; then the angle ABC conteined under the adjoined lines AB, CB, is said to be an angle in a segment.

VIII. But when the right lines AB, BC comprehending the angle ABC, do receive any periphery of the circle ADC, then the angle ABC is said to stand upon that periphery.

IX.

IX. A sector of a circle (ADB) is when an angle **A D B** is set at the centre D of that circle; namely, that figure ADB, comprehended under the right lines AD, BD conteining the angle, and the part of the circumference received by them **AB**.

X. Like segments of a circle (**AB C**, D E **F**) are those which conclude equall angles (AB**C**, DEF;) or, in whom the angles ABC, DE**F** are equall.

PROP. I.

To find the centre F *of a circle given* ABC.

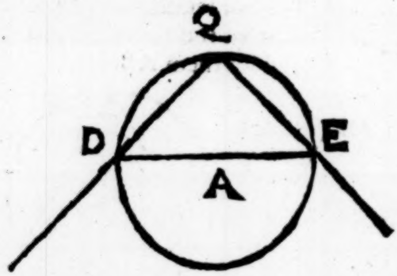
Draw a right line A **C** any-wise in the circle, which bisect in E, through E draw a perpendicular **D**B, and bisect the same in **F**; the point F shall be the center.

If you deny it, let G a point without the line DB be the centre (for it cannot be in the line BD, since that cannot be divided equally in any point but F;) let the lines G A, G C, G E be drawn. Now if G be the centre, *a* then is GA = GC, and AE = EC by construction, and the side GE common. *b* Therefore are the angles GEA, GEC equall, and *c* consequently right. *d* Therefore the angle GEC = FEC. *e Which is absurd.*

a 15. *def.* 1.
b 8. 1.
c 10. *def.* 1.
d 12. *ax.*
e 9. *ax.*

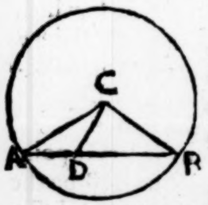
Coroll.

Hence , If a right line B D bisect any right line AC in a circle at right angles , the centre shall be in the line BD that cuts the other.



Andr. Tacq. *The centre of a circle is easily found out by applying the top of a Square to the circumference thereof.* For if the right line D E that joins the points D , E , in which the sides of the Square Q D, Q E cut the circumference , be bisected in A, the point A shall be the centre. The demonstration whereof depends upon *Prop.* 31. of this Book.

P R O P. II.



If in the circumference of a circle CAB *any two points* A , B *be taken, the right line* AB *which joins those two points shall fall within the circle.*

Take in the right line AB any point D; from the center C draw CA, CD, CB. Because C A a = C B, therefore is the angle A b = B. But the angle CDB c $\sqsubset$ A, therefore is C D B $\sqsubset$ B. therefore C B d $\sqsubset$ C D. But CB onely reaches the circumference , therefore CD comes not so far; wherefore the point D is within the circle. The same may be proved of any other point in the line AB. And therefore the whole line AB fals within the circle. *Which was to be Dem.*

a 15. d f. 1.
b 5. 1.
c 16. 1.
d 19. 1.

Co-

Coroll.

Hence, If a right line touch a circle, so that it cut it not, it touches but in one point.

PROP. III.

If in a circle E A B C, *a right line* B D *drawn through the centre, bisect any other line* A G *not drawn through the centre, it shall also cut it at right angles: And if it cuts it at right angles, it shall also bisect the same.*

From the centre E let the lines E A, E C be drawn.

1. *Hyp.* Because A F a = F C, and E A b = EC, and the side EF common; the angles EFA, EFC c shall be equall, and d consequently right. *Which was to be Dem.* — a hyp. b 15.def.1. c 8. 1. d 10.def.1.

Hyp. 2. Because E F A e = E F C, and the angle EAF f = ECF, and the side E F common; g therefore is AF = FC. Therefore AC is cut into two equall parts. *Which was to be Dem.* — e hyp and 12.ax. f 5.1. g 26. 1.

Coroll.

Hence, In any equilateral or Isosceles triangle, if a line drawn from the verticall angle bisect the base, that line is perpendicular to it. And on the contrary, a perpendicular drawn from the verticall angle bisects the base.

PROP. IV.

If in a circle A C D *two right lines* AB, CD *cut through one another, yet neither of them passe through the centre* E, *then neither of those lines are divided into equall parts.*

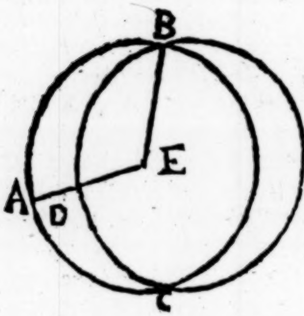
For if one line passe through

the centre, it appears that it cannot be bisected by the other; because by Hypothesis, the other does not passe through the centre.

If neither of them passe through the centre, then from the centre E draw E F: now if A B, C D were both bisected in F, then [a] would the angles EFB, EFD be both right, and consequently equall [b] *Which is absurd.*

[a] 3. 3.
[b] 9. ax.

PROP. V.



If two circles BAC, BDC *cut one the other, they shall not have the same centre* E.

For otherwise the lines EB, EDA drawn from E the common centre, would D E be [a] = E B [a] = E A. [b] *Which is absurd.*

[a] 15. def. 1.
[b] 9. ax.

PROP. VI.

If two circles BAC, BDC, *inwardly touch one the other* (in B) *they have not one and the same centre* F.

For otherwise the right lines F B, F DA drawn from the centre F, would be F D [a] = F B [a] = FA. [b] *Which is absurd.*

[a] 15. def. 1.
[b] 9. ax.

PROP.

PROP. VII.

If in A B *the diameter of a circle some point* G *be taken, which is not the centre of the circle, and from that point certain right lines* GC, GD, GE *fall on the circle, the greatest line shall be that* (GA) *in which is the centre* F; *the least, the remainder of the same line* (GB.) *And of all the other lines, the line* G C *nearest to that which was drawn through the centre is alwayes greater then any line farther removed* G D; *and only two lines are equall* GE, GH, *which fall upon the circle from the same point, on each side of the least* G B *or of the greatest* GA.

From the centre F draw the right lines FC, FD, FE; and *a* make the angle BFH = BFE. a 23. 1.

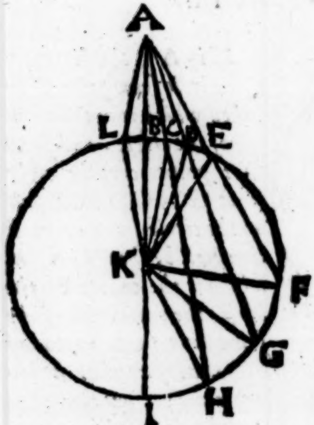
1. GF + FC (that is GA) *a* $\sqsubset$ GC. *Which was to be Dem.* a 20. 1.

2. The side FG is common, and FC *b* = FD, and the angle GFC *c* $\sqsubset$ GFD; *d* wherefore the base GC $\sqsubset$ GD. b 15. *def.* 1. c 9. *ax.* d 24. 1.

3. FB (FE) *e* $\sqsupset$ GE+GF. Therefore FG, which is common, being taken away from both, there remains BG $\sqsupset$ EG. e 20. f 5. *ax.*

4. The side FG is common, and FE = FH, and the angle BFH *g* = BFE; *h* Therefore is GE = GH. But that no other line G D from the point G can be equall to GE, or GH, is already proved. *Which was to be Demonstrated.* g *const* h 4. 1.

PROP. VIII.



If some point A *be taken without a circle, and from that point be drawn certain right lines* AI, AH, AG, AF *to the circle, & of those one* A I *be drawn through the centre* K, *and the others anywise; of all those lines that fall on the concave of the circumference, that is the greatest* A I *which is drawn through the centre; and of the others, that which is nearest* (AH) *to the line that passes through the centre is greater then that which is more distant* A G. *But of all those lines that fall on the convexe part of the circle, the least is that* AB *which is drawn from the point* A *to the diameter* I B; *and of the others, that* (A C) *which is nearest to the least, is lesse then that which is farther distant* AD. *And from that point there can be onely two equall right lines* AC, AL *drawn, which shall fall on the circumference on each side of the least line* A B *or of the greatest* AI.

From the centre K draw the right lines KH, KG, KF, KC, KD, KE. and make the angle A K L $=$ AKC.

a 20. 1. 1. AI (AK+KH) *a* $\sqsubset$ AH.

2. The side AK is common, and KH$=$KG, and
b 24. 1. the angle AKH $\sqsubset$ AKG; *b* therefore the base AH $\sqsubset$ AG.

c 20. 1. 3. KA *c* $\sqsupset$ KC + CA. From hence take away
d 5. ax. KC, KB that are equall; then will remain AC *d* $\sqsupset$ AC.

4. AC + CK *e* $\sqsupset$ AD+DK. From thence take
e 21. 1. away CK, DK that are equall; then remains AC
f 5. ax. *f* $\sqsupset$ AD.

5. The

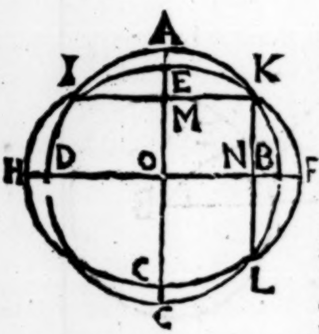
5. The side KA is common, and KL = KC, and the angle AKL g = AKC; h therefore KA = CA. But that no other line could be drawn equall to these, was proved above. Therefore, &c. g *const.* h 4. 1.

PROP. IX.

If in a circle BCK *a point* A *be taken, and from that point more then two equall right lines* AB, AC, AK, *drawn to the circumference, then is that point* A *the centre of the circle.*

For a from no point without the centre can more then two right lines equall be drawn to the circumference. Therefore A is the centre. *Which was to be Dem.* a 7. 3.

PROP. X.

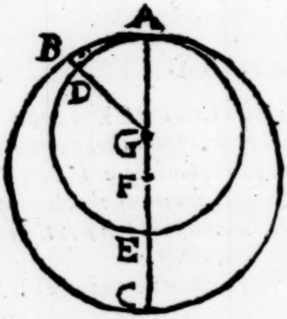


A circle IAKBL *cannot cut another circle* IEKFL *in more then two points.*

Let one circle, if it may be, cut the other in three points I, K, L. and IK, KL being join'd, let them be bisected in M and N. a Both circles have their centres in their perpendiculars MC, NH, and in the intersection of those perpendiculars which is O. Therefore the circles that cut each other have the same centre. Which is false, by *Prop.* 5. 3. a *cor.* 1. 3.

PROP.

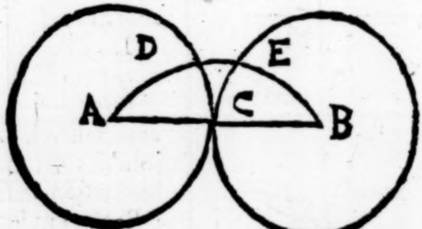
PROP. XI.



If two circles GADE, FABC *touch one the other inwardly, and their centres be taken* G, F; *a right line* FG *joyning their centres, and produced, shall cut the circumference in* A *the point of contact of the circles.*

If it can be, let the right line FG produced cut the circles in some other point then A. so that not FGA, but FGDB shall be a right line. Let
a 15 def 1. the line GA be drawn. Now, because GD a $=$ GA, &
b 7. 3. GB b $\sqsupset$ GA (since the right line FGB passes through F the centre of the greater circle) therefore
c 9. ax. is GB $\sqsupset$ GD. c *Which is absurd.*

PROP. XII.

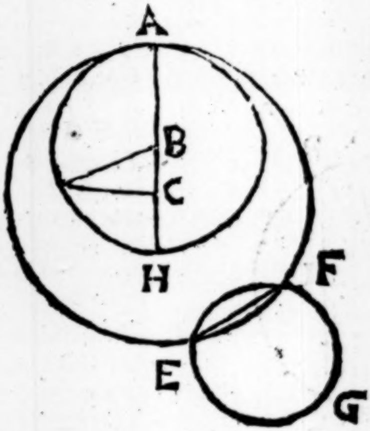


If two circles ACD, BCE *touch one the other outwardly, the right line* AB *which joins their centres* A, B, *shall passe through the point of contact* C.

If it may be, let ADEB be a right line cutting the circles not in the point of contact C, but in the points D, E; draw AC, CB. then is AD + EB (AC
a 20. 1.
b 9 ax. + CB) a $\sqsubset$ ADEB. b *Which is absurd.*

PROP.

PROP. XIII.



A circle CAF *cannot touch a circle* BAH *in more points then one* A, *whether it be inwardly or outwardly.*

1. Let one circle (if it can be) touch another in two points A, H. *a* Then will the right line CB that joins the centres, if it be produced, fall as well in A, as H. Now because CH *b* = CA, and BH ⊏ CH, therefore is BA (*c* BH) ⊏ CA. *d Which is absurd.* a 11. 3. b 15. def 1. c 15. def. 1. d 9. ax.

2. If it be said to touch outwardly in the points E and F, then draw the line EF, *e* which will be in both circles. Therefore those circles cut one the other; Which is against the Hyp. e 2. 3.

PROP. XIV.

In a circle EABC *equall right lines* AC, BD *are equally distant from the centre* E: *and right lines* AC, BD *which are equally distant from the centre, are equall among themselves.*

From the centre E draw the perpendiculars EF, EG. *a* which will bisect the lines AC, BD. join EA, EB. a 3. 3.

1. *Hyp.* AC = BD. therefore AF *b* = BG. But also b 7. ax.

c 47.1. and 3. ax. so EA = EB. therefore FEq *c* = EAq − AFq = EBq − BGq *c* = EGq. *d* Therefore FE = EG.

d schol 48. 1. 2. *Hyp.* EF = EG. Therefore AFq *e* = EAq − EFq = EBq − EGq = BGq. Therefore AF *d* = GB, and

e 6. ax. *e* consequently AC = BD. *Which was to be Dem.*

PROP. XV.

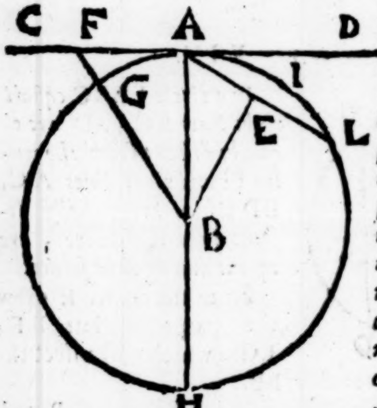
In a circle G A B C *the greatest line is* A D *the diameter; and of all other lines, that line* F E, *which is nearest to the centre* G *is greater then any line* B C *farther distant from it.*

1. Draw GB, and GC. The diameter AD (*a* GB + GC) *b* ⊏ BC.

a 15. def. 1. b 20. 1. 2. Let the distance GI be ⊏ GH. Take GN = GH. Through the point N draw KL perpendicularly to GI; join GK, GL. Because GK = GB, and GL = GC, and the angle KGL

c 24. 1. ⊏ BGC; *c* therefore is KL (FE) ⊏ BC. *Which was to be Dem.*

PROP. XVI.



A line CD *drawn from the extreme point of the diameter* HA *of a circle* BALH, *perpendicular to the said diameter, shall fall without the circle; and between the same right line and the circumference cannot be drawn another line* AL. *And the angle of the*

the semicircle BAI, *is greater then any right-lined acute angle* BAL; *and the remaining angle without the circumference* DAI *is lesse then any right-lined angle.*

1. From the centre B to any point F in the right line AC, draw the rigt line BF. The side BF subtending the right angle B A F is [a] greater then the side B A which is opposite to the acute angle BFA. Therefore whereas BA (BG) reaches to the circumference, BF shall reach further; and so the point F. and for the same reason is any other point of the line AC placed without the circle. a 19. 1.

2. Draw BE perpendicular to A L. The side BA opposite to the right angle B E A is [b] greater then the side BE which subtends the acute angle BAE; therefore the point E, and so the whole line EA, fals within the circle. b 19. 1.

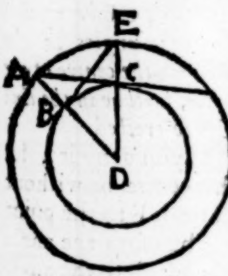
3. Hence it follows that any acute angle, to wit, E AD, is greater then the angle of contact DAI. and that any acute angle BAL is lesse then the angle of a semicircle BAI. *Which was to be Dem.*

Coroll.

Hence, A right line drawn from the extremity of the diameter of a circle, and at right angles, is a tangent to the said circle.

From this proposition are gathered many paradox and wonderfull consectaries, which you may meet with in the interpreters.

PROP. XVII.



From a point given A *to draw a right line* AG *which shall touch a circle given* DBC.

From D the centre of the circle given let a line D A cutting the circumference in B, be drawn to the point given A; from the centre D describe another circle through the point A; and from B draw a perpendicular to A D, which shall meet with the circle

circle AB in the point E; and draw ED meeting with the circle BC in the point C. Then the line drawn from A to C shall touch the circle DBC.

a 15 def. 1. b 4. 1. For DB a = DC, and DE a = DA, and the angle D is common; b therefore the angle ACD = EBD and right. c Therefore AC touches the circle in C. *Which was to be Done.*

c cor. 16. 3.

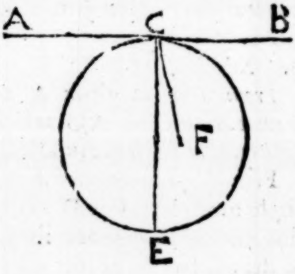
PROP. XVIII.

If any right line AB *touch a circle* FEDC, *and from the centre to the point of contact* E *a right line* FE *be drawn; that line* FE *shall be perpendicular to the tangent* AB.

If you deny it, let some other line FG be drawn from the centre F perpendicular to the tangent, and a cutting the circle in D. Therefore, whereas the angle FGE is said to be right, b thence is the angle FEG acute; c so that FE (FD) ⊏ FG. d *Which is absurd.*

a 2. def. 3. b cor. 17. 1. c 19. 1. d 9. ax.

PROP. XIX.



If any right line AB *touch a circle, and from the point of contact* C *a right line* CE *be erected at right angles to the tangent, the centre of the circle shall be in the line* CE *so erected.*

If you deny it, let the centre be without the line CE in the point F; and from F to the point of contact let FC be drawn. Therefore the angle FCB is right, and a consequently equall to the angle ECB, which was right by Hypothesis. b *Which is absurd.*

a 12. ax. b 9 ax.

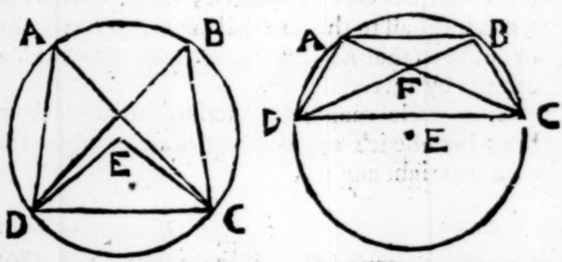
PROP.

PROP. XX.

In a circle DABC, *the angle* BDC *at the centre is double of the angle* BAC *at the circumference, when the same arch of the circle* BC *is the base of the angles.*

Draw the diameter ADE. The outward angle
BDE *a* = DAB + DBA *b* = 2 DAB. Likewise the a 32. 1. b 5. 1.
angle EDC = 2 DAC. Therefore in the first case
the whole angle BDC = 2 BAC. and in the third
case the *c* remaining angle BDC = 2 BAC. *Which* c 10. ax.
was to be Dem.

PROP. XXI.



In a circle EDAC, *the angles* DAC *and* DBC *which are in the same segment, are equall one to the other.*

1. *Case.* If the segment DABC be greater then a semicircle, from the centre E draw ED, EC. Then is twice the angle A *a* = E *a* = 2 B. *W. W. to be Dem.*

2. *Case.* If the segment be lesse then a semicircle, a 20. 3.
then is the sum of the angles of the triangle ADF
equall to the sum of the angles of the triangle BCF.
from each let AFD be taken away *b* equall to b 15. 1.
BFC,

e by the 1. ax. BFC, and ADB c = ACB be likewise taken away, then remains DAC = DBC. *W.W. to be Dem.*

PROP. XXII.

The angles ADC, ABC *of a quadrilateral figure* ABCD *described in a circle, which are opposite one to the other, are equall to two right angles.*

Draw AC, BD. The angle ABC + BCA + BAC a = 2 right. But BDA b = BCA, and BDC b = BAC. c Therefore ABC + ADC = 2 right angles. *Which was to be Dem.*

a 32. 1.
b 21. 3.
c 1. ax.

Coroll.

1. Hence, If one side * AB of a quadrilateral described in a circle be produced, the externall angle EBC is equall to the internall angle ADC, which is opposite to that ABC which is adjacent to EBC. as appears by 13. 1. and 3. *ax.*

* *See the following Diagr.*

2. A circle cannot be described about a Rhombus; because it's opposite angles are greater or lesse then two right angles.

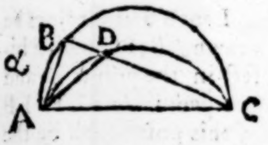
Schol.

If in a quadrilateral ABCD *the angles* A *and* C, *which are opposite, be equall to two right, then a circle may be described about that quadrilateral.*

For a circle will passe through any 3 angles (as shall appear by 5. 4.) I say that shall passe through A the fourth also of such

a quadrilaterall: For if you deny it, let the circle passe through F. Therefore the right lines BF, FD, BD being drawn, the angle C+F a = 2 right b = C+A wherefore, A c is equall to F. d *Which is absurd.* a 22. 3. b hyp. c 3. ax. d 21. 1.

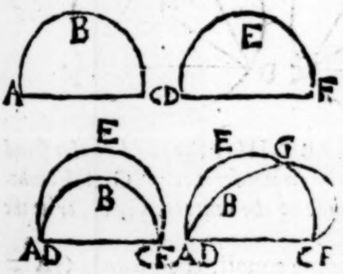
Prop. XXIII.



Two like and unequall segments of circles ABC, ADC, *cannot be set on the same right line* AC, *and the same side thereof.*

For if they are said to be like, draw the line C B cutting the circumferences in D and B, join A B and A D. Because the segments are supposed like, a therefore is the angle ADC = ABC. b *Which is absurd.* a 10 def 3. b 16. 1.

Prop. XXIV.



Like segments of circles A B C, DEF *upon equall right lines* A C, DF, *are equall one to the other.*

The base A C being laid on the base D F will agree with it, because AC=DF. Therefore the segment A B C shall agree with the segment DEF (for otherwise it shall fall either within or without, and if so a then the segments are not like, which is contrary to the Hypothesis, and at least it shall fall partly within and partly without, and so cut in 3 points, b which is absurd. c Therefore the segment ABC = DEF. *Which was to be Dem.* a 23. 3. b 10. 3. c 8 ax.

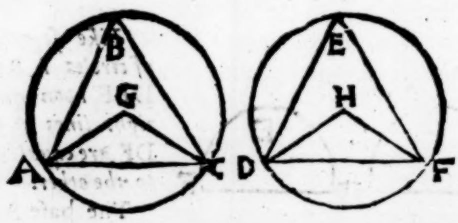
Prop. XXV.

A segment of a circle ABC *being given, to describe the whole circle whereof that is a segment.*

Let two right lines be drawn AB, BC, which bisect in the points D and E. From D and E draw the perpendiculars DF, EF meeting in the point F. I say this point shall be the centre of the circle.

a cor. 1. 3. For the centre shall be as well in a DF as EF, therefore it must be in the common point F. *Which was to be Done.*

Prop. XXVI.



In equall circles GABC, HDEF, *equall angles stand upon equall parts of the circumference*, AC, DF; *whether those angles be made at the centres*, G, H, *or at the circumferences*, B, E.

Because the circles are equall, therefore is GA = HD, and GC = HF; also by Hypothesis the angle G = H; a therefore AC = DF. Moreover the angle B b = $\frac{1}{2}$ G c = $\frac{1}{2}$ H b = E. d Therefore the segments ABC, D E F are like, and e consequently equall. f whence the remaining segments also AC, DF are equall. *Which was to be Dem.*

a 4. 1.
b 20. 3.
c hyp.
d 10. def. 3.
e 24. 3.
f 3. ax.

Schol.

Schol.

In a circle ABCD let an arch AB be equall to DC; then shall AD be parallel to BC. For the right line AC being drawn, the angle ACB a = CAD; wherefore by 27. 1. the said sides are parallel. a 16. 3.

PROP. XXVII.

In equall circles GABC, HDEF, *the angles standing upon equall parts of the circumference* AC, DF, *are equall between themselves, whether they be made at the centres* G, H, *or at the circumferences,* B, E.

For if it be possible, let one of the angles AGC be ⊏ DHF, and make AGI = DFH. thence is the arch AI a = DF b = AC. c *Which is absurd.* a 16 3. b hyp. c 9. ax.

Schol.

A right line EF, *which being drawn from* A *the middle point of any periphery* BC, *toucheth the circle, is parallel to the right line* BC *subtending the said periphery.*

From the centre D draw a right line DA to the point of contact A, and join DB, DC.

The side DG is common, and DB = DC, and the angle BDA a = CDA, (because the arches BA, a 27 3

b hyp. c 4. 1. d 10. def. 1. e hyp. f 28. 1. CA are *b* equall) therefore the angles at the base DGB, DGC are *c* equall, and *d* consequently right; But the inward angles GAE, GAF are also *e* right, *f* therefore BC, EF are parallel. *Which was to be Dem.*

PROP. XXVIII.

In equall circles GABC, HDEF, *equall right lines* AC, DF *cut off equall parts of the circumference, the greatest* ABC *equall to the greatest* DEF, *and the least* AIC *to the least* DKF.

From the centres G, H draw GA GC, & HD, HF.

a hyp. b 28 1. c 6. 3. d 3. ax. Because GA = HD, and GC = HF, and AC *a* = DF, *b* therefore is the angle G = H; *c* whence the arch AIC = DKF; *d* and so the remaining arch ABC = DEF. *Which was to be Dem.*

But if the subtended line AC be ⊏ or ⊐ then DF, then in like manner will the arch AC be ⊏ or ⊐ then DF.

PROP. XXIX.

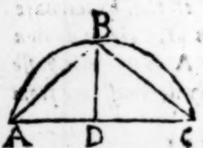
In equall circles GABC, HDEF, *equall right lines* AC, DF *subtend equall peripheries* ABC, DEF.

a hyp. b 27 3. c 4. 1. Draw the lines GA, GC, and HD, HF. Because GA = HD, and GC = HF, and (because the arches AC, DF are *a* equall) the angle G *b* = H. *c* therefore is the base AC = DF. *Which was to be Dem.*

This and the three precedent propositions may be understood also of the same circle.

PROP.

PROP. XXX.

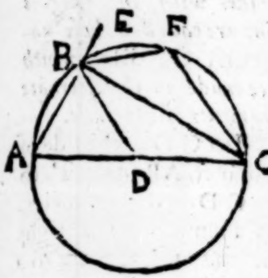


To cut a periphery given ABC *into two equall parts.*

Draw the right line AC, & bisect it in D; from D draw a perpendicular DB meeting with the arch in B, it shall bisect the same.

For join AB, and CB. The side DB is common, and AD a = DC, and the angle ADB b = CDB. a conſt. b 12 ax.
c therefore AB = BC; d whence the arch AB = BC. c 4. 1. d 28 3.
Which was to be Done.

PROP. XXXI.



In a circle the angle ABC, *which is in the semi-circle, is a right angle; but the angle, which is in the greater segment* BAC *is lesse then a right angle, and the angle which is in the lesser segment* BFC *is greater then a right angle. Moreover, the angle of the greater segment is greater then a right angle, and the angle of the lesser segment is lesse then a right angle.*

From the centre D draw DB. Because DB = DA, therefore is the angle A a = DBA, and a 5 1.
the angle DCB a = DBC, b therefore the angle b 2. ax.
ADC = A + ACB c = EBC, d so that ABC and c 32. 1. d 10 def. 1.
EBC are right angles. *W. W. to be Dem.* e Therefore e cor. 17. 1.
BAC is an acute angle. f *W. W. to be Dem.* And further, f 22. 3.
whereas BAC + BFC = 2 right, therefore BFC is an obtuse angle. Lastly the angle conteined under the right line CB, and the arch BAC is greater then the right angle ABC; but the angle made by the right line CB and the periphery of the lesser segment BFC g is lesse then the right angle ABC. *Which was to be Dem.* g 9. ax.

Schol.

In a right-angled triangle ABC, *if the hypotenuse (or subtended line)* AC *be bisected in* D, *a circle drawn from the centre* D *through the point* A *shall also passe through the point* B: *As you may easily demonstrate from this* prop. *and* 21. 1.

PROP. XXXII.

If a right line AB *touch a circle, and from the point of contact be drawn a right line* CE *cutting the circle, the angles* ECB, ECA *which it makes with the tangent line are equall to those angles* EDC, EFC *which are made in the alternate segments of the circle.*

a 26 3.
b 19. 3.
c 31. 3.
d 32. 1.
e constr.
f 3 ax.

Let CD, the side of the angle EDC, be perpendicular to AB (a for it's to the same purpose) b therefore CD is the diameter. c therefore the angle CED in a semicircle is a right angle, d and therefore the angle D + DCE = to a right angle e = ECB + DCE. f Therefore the angle D = ECB. *Which was to be Dem.*

g 13. 1.
h 22. 3.
k 3. ax.

Now whereas the angle ECB + ECA g = 2 right h = D + F, from both of these take away ECB and D, which are equall, k then remains ECA = F. *Which was to be Dem.*

PROP. XXXIII.

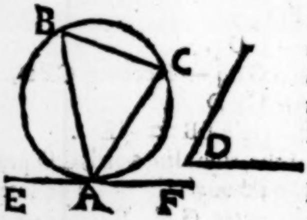
Vpon a right line AB *to describe a segment of a circle* AIEB *which shall contein an angle* AIB *equall to a right-lined angle given* C.

a 23. 1.

a Make the angle

angle BAD = C. Through the point A draw the line AE perpendicular to HD. At one end of the line given AB make an angle ABF = BAF, one side whereof let it cut the line AE in F; from the centre F through the point A, describe a circle which shall passe through B (because the angles FBA *b* = FAB, and *c* therefore FB = FA.) AIB is the segment sought. For because HD is perpendicular to the diameter AE, it therefore *d* touches the circle HD which AB cuts. And therefore the angle AIB *e* = BAD *f* = C. *Which was to be done.* b constr. c 6. 1. d cor 16 3. e 32 3. f constr.

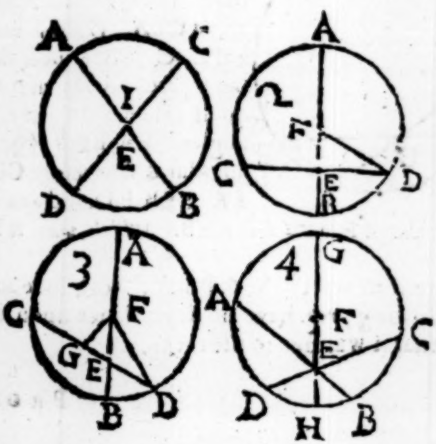
PROP. XXXIV.



From a circle given ABC *to cut off a segment* ABC *conteining an angle* B *equall to a right-lined angle given* D.

a Draw a right line EF which shall touch the circle given in A. *b* let AC be drawn also making an angle FAC = D. This line shall cut of ABC conteining an angle B *c* = CAF *d* = D. *Which was to be Done.* a 17. 3. b 23. 1 c 32. 3. d constr

PROP. XXXV.



If in a circle FBCA *two right lines* AB, DC *cut each other, the rectangle comprehended under the segments* AE, EB *of the one,*

one, shall be equall to the rectangle comprehended under the segments CE, ED *of the other.*

1. *Case.* If the right lines cut one the other in the centre, the thing is evident.

2. *Case.* If one line AB passe through the centre F, and bisect the other line CD, then draw FD. Now

a 5. 2.
b sch. 48. 1.
c 47. 1.
d hyp.
e 3. ax.

the rectangle AEB + FEq [a] = FBq [b] = FDq [c] = EDq + FEq [d] = CED + FEq. [e] Therefore the rectangle AEB = CED. *Which was to be Dem.*

3. *Case.* If one of the lines AB be the diameter, and cut the other line CD unequally, bisect CD by FG a perpendicular from the centre.

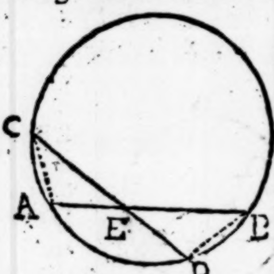
f 5. 2.
g 47. 1.
h 5. 2.
k 47. 1.

These are equall. { The rectangle AEB + FEq. / [f] FBq (FDq.) / [g] FGq + GDq. / FGq + [h] GEq + Rectang. CED, / [k] FEq + CED. }

l 3. ax.

[l] Therefore the rectangle AEB = CED.

4. *Case.* If neither of the right-lines AB, CD passe through the centre, then through the point of intersection E, draw the diameter GH. By that which hath been already demonstrated, it appears that the rectangle AEB = GEH = CED. *W.W. to be Dem.*



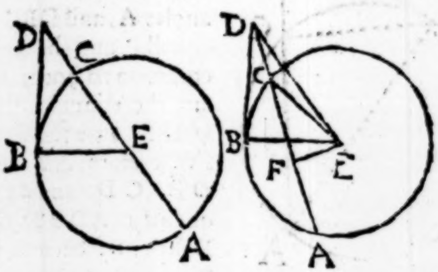
a 15. 1.
b 21. 3.
c cor. 32. 1.
d 4. 6.
e 16. 6.

More easily, and generally, thus; Join AC and BD, Then because the angles [a] CEA, DEB, and [b] also C, B (upon the same arch AD) are equall, thence are the triangles CEA, BED [c] equiangular. [d] Wherefore CE, EA :: EB, ED. and [e] consequently CE x ED = AE x EB. *Which was to be Dem.*

The citations out of the 6. Book, both here and in the following *prop.* have no dependance upon the same; so that it was free to use them.

PROP.

PROP. XXXVI.



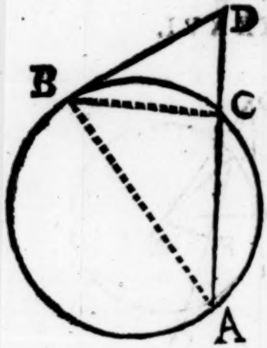
If any point D *be taken without a circle* EBC, *and from that point two right lines* DA , DB *fall upon the circle , whereof one* D A *cut the circle , the other* D B *touches it , the rectangle comprehended under the whole line* DA *that cuts the circle , and under* DC *that part which is taken from the point given* D *to the convexe of the periphery , shall be equall to the square made of the tangent line.*

1.*Case.* If the secant AD passe through the centre, then join EB, this *a* will make a right angle with the line DB, wherefore $DBq +$ *d* $EBq (ECq)$ *b* $= EDq$ *c* $= AD \times DC + ECq$. Therefore $AD \times DC = DBq$. *Which was to be Dem.* a 18. 3. b 47. 1. c 6. 2. d 3. ax.

2.*Case.* But if AD passe not through the centre then draw EC, EB, ED , and EF perpendicular to AD, *a* wherefore AC is bisected in F.

Because $BDq + EBq$ *b* $= DEq$ *b* $= EFq + FDq$ *c* $= EFq + ADC + FCq$ *d* $= ADC + CEq (EBq.)$ *e* Therefore is $BDq = ADC$. *Which was to be Dem.* a 3. 3. b 47. 1. c 6. 2. d 47. 1. e 3. ax.

More



More easily, and generally, thus; Draw AB and B C. Then because the
a 31. 3. angles A, and DBC *a* are equall, and the angle D common to both, thence are the triangles BDC,
b 32. 1. ∀DB *b* equiangular.
c 4 6 *c* Wherefore AD. DB ::
d 17. 6. D B. C D. and *d* consequently AD x D C = DBq. *Which was to be Dem.*

Coroll.

1. Hence, If from any point A taken without a circle, there be severall lines AB, AC drawn which cut the circle, the rectangles comprehended under the whole lines AB, AC, and the outward parts AE, AF are equall between themselves.

For if the tangent A D be
a 36. 3. drawn, then is CAF=ADq *a* = BAE.

2. It appears also from hence, that if two lines AB, AC drawn from the same point do touch a circle, those two lines are equall one to the other.

a 36. 3. For if AE be drawn cutting the circle, then is ABq *a* =
b 36. 3. EAF *b* = ACq.

3. It

3. It is also evident that from a point A taken without a circle, there can be drawn but two lines AB, AC that shall touch the circle.

For if a third line AD be said to touch the circle, thence is AD *c* = AB *c* = AC. *d Which is absurd.* c 2 cor. d 8. 3.

4. And on the contrary, it is plain, that if two equall right lines AB, AC fall from any point A upon the convexe periphery of a circle, and that if one of these equall lines AB touch the circle, then the other AC touches the circle also.

For if possible, let not AC, but another line AD, touch the circle; therefore is AD *e* = AC *f* = AB. *g Which is absurd.* e 2. cor. f hyp. g 8. 3.

P R O P. XXXVIII.

If without a circle EBF *any point* D *be taken, and from that point two right lines* DA, DB *fall on the circle, whereof one line* DA *cuts the circle, the other* DB *falls upon it; and if also the rectangle comprehended under the whole line that cuts the circle, and under that part of it* DC *which is taken betwixt the point* D *and the convexe periphery, be equall to that square which is made of the line* DB *falling on the circle, I say that that line* DB *so falling shall touch the circle given.*

From the point D *a* let a tangent DF be drawn, and from the centre E draw ED, EB, EF. Now because DBq *b* = ADC *c* = DFq, therefore is DB *d* = DF: But EB = EF, and the side ED common; *e* therefore the angle EBD = EFD. but EFD is a right angle, and *f* therefore EBD is right also. and *g* therefore DB touches the circle. *W. W. to be Dem.* a 17. 3. b hyp. c 36. 3. d 1. ax. & sch. 47. 1. e 8. 1. f 12. ax g cor. 16. 3

Coroll.

From hence it follows that the *h* angle EDB = EDF. h 8. 1.

The End of the third Book.

THE

OF EUCLIDE'S ELEMENTS.

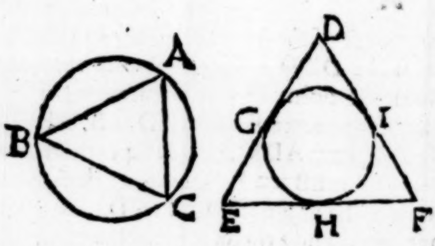
Definitions.

I. A Right-lined figure is said to be inscribed in a right-lined figure, when every one of the angles of the inscribed figure touch every one of the sides of the figure wherein it is inscribed.

So the triangle DEF *is inscribed in the triangle* ABC.

II. In like manner a figure is said to be described about a figure, when every one of the sides of the figure circumscribed touch every one of the angles of the figure about which it is circumscribed.

So the triangle ABC *is described about the triangle* DEF.



III. A right-lined figure is said to be inscribed in a circle, when all the angles of that figure which is inscribed do touch the circumference of the circle.

IV. A right-lined figure is said to be described about a circle, when all the sides of the figure which

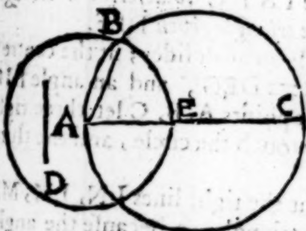
which is circumscribed touch the periphery of the circle.

V. After the like manner a circle is said to be inscribed in a right-lined figure, when the periphery of the circle touches all the sides of the figure in which it is inscribed.

VI. A circle is said to be described about a figure when the periphery of the circle touches all the angles of the figure, which it circumscribes.

VII. A right line is said to be coapted or applyed in a circle when the extremes thereof fall upon the circumference; *as the right line* AB.

Prop. I. *Probl.* 1.

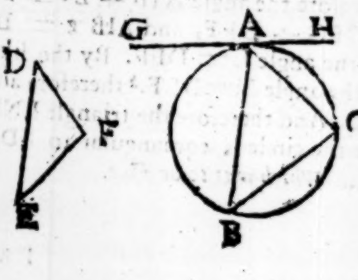


In a circle given ABC *to apply a right line* AB *equall to a right line given* D, *which doth not exceed* AC *the diameter of the circle.*

From the centre A by the space AE=D a describe a circle meeting with the circle given in B. draw AB, Then is AB b=AE c=D. *W.W. to be Done.*

a 3 *post.* and 3. 1.
b 15 *def.* 1.
c *constr.*

Prop. II. *Probl.* 2.



In a circle given ABC *to describe a triangle* ABC, *equiangular to a triangle given* DEF.

Let the right

a 17. 3. b 23. 1. right line GH *a* touch the circle given in A; *b* make the angle HAC = E, *b* and the angle GAB = F. then join BC; and the thing is done.

c 32. 3. d *constr.* e 32. 1. For the angle B *c* = HAC *d* = E, and the angle C *e* = GAB *d* = F; *e* whence also the angle BAC = D. Therefore the triangle BAC inscribed in the circle is equiangular to DEF. *Which was to be done.*

PROP. III.

About a circle given IABC *to describe a triangle* LNM *equiangular to a triangle given* DEF.

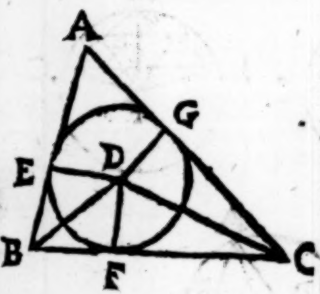
a 23. 1. b 17. 3. Produce the side EF on both sides; at the centre I *a* make an angle AIB = DEG, and an angle BIC = DFH. Then in the points A, B, C let three right lines LN, LM, NM *b* touch the circle, and the thing is done.

c 13. ax. d 18. 3. For it's evident that the right lines LN, LM, MN will meet and make a triangle, *c* because the angles LAI, LBI are right; so that the *d* right line AB produced will make the angles LAB, LBA, lesse then 2 right angles.

e *schol.* 32. 1. f 13. 1 g *constr.* h 3 ax. k 32. 1. Since, therefore the angle AIB + L *e* = 2 right angles *f* = DEG + DEF, and AIB *g* = DEG; *h* therefore is the angle L = DEF. By the like way of argument the angle M = DEF. *k* therefore also the angle N = D. And therefore the triangle LNM described about the circle is equiangular to EDF the triangle given. *Which was to be done.*

PROP.

PROP. IV.



In a triangle given ABC, *to describe a circle* EFG.

a Bisect the angles B and C with the right lines BD, CD meeting in the point D, b & draw the perpendiculars DE, DF, DG. A circle described from the centre D through E, will passe through G and F, and touch the three sides of the triangle.

a 9. 1.

b 12. 1.

For the angle DBE c = DBF; and the angle DEB d = DFB; and the side DB common. e therefore DE = DF. By the like argument DG = DE. The circle therefore described from the centre D passes through the 3 points E, F, G. and whereas the angles at E, F, G are right, therefore it touches all the sides of the triangle. *Which was to be done.*

c constr.
d 12. ax.
e. 26. 1.

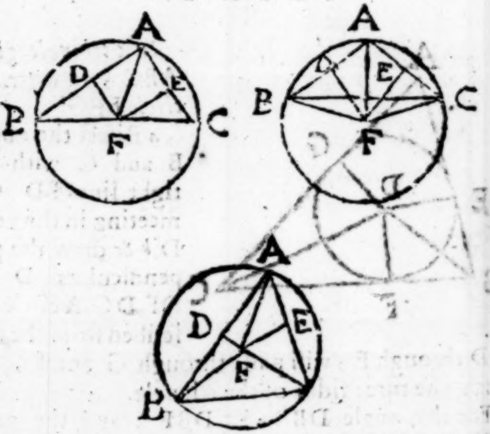
Schol.

Hence, *The sides of a triangle being known, their segments which are made by the touchings of the circle inscribed shall be found,* Thus; Petr. Herig.

Let AB be 12, AC 18, BC 16. then is AB + BC = 28. Out of which subduct 18 = AC = AE + FC, then remains 10 = BE + BF. Therefore BE, or BF = 5; and consequently FC, or CG, = 11. Wherefore GA, or AE, = 7.

PROP.

P R O P. V.



About a triangle given A B C *to describe a circle* FABC.

a 10 & 11.1. *a* Bisect any two sides BA, CA with perpendiculars DF, EF meeting in the point F. I say this shall be the centre of the circle.

For, let the right lines FA, FB, FC be drawn.
b constr. Now because AD *b* = DB and the side DF com-
c constr. and 12. ax. mon, and the angles FDA *c* = FDB, therefore is
d 4. 1. FB *d* = FA. After the same manner is FC = FA. Therefore a circle described from the centre F shall passe through the angles of the triangle given (*viz.*) B, A, C. *Which was to be done.*

Coroll.

* 31. 3. * Hence, if a triangle be acute-angled, the centre shall fall within the triangle; if right-angled, in the side opposite to the right angle, & if obtuse-angled, without the triangle.

Schol.

By the same method may a circle be described, that shall passe through 3 points given, not being in the same strait line.

PROP.

PROP. VI.

In a circle given EABCD *to inscribe a square* ABCD.

a Draw the diameters AC, BD cutting each other at right angles in the centre E. Join the extremes of these diameters with the right lines AB, BC, CD, DA. *And the thing is done.* a 11. 1.

Now because the 4 angles at E are right, the b arches and c subtended lines AB, BC, CD, DA are equall; therefore is the figure ABCD equilateral, and all the angles in semicircles, and so d right. e Therefore ABCD is a square inscribed in a circle given. *Which was to be done.* b 26. 3. c 29. 3. d 31. 3. e 29 3. def. 1.

PROP. VII.

About a circle given EABCD *to describe a square* FHIG.

Draw the diameters AC, BD cutting one the other at right angles; through the extremes of these diameters a draw tangents meeting in F, H, I, G then I say it's done. a 17. 3.

For because b the angles A and C are right, c therefore is FG parallel to HI. After the same manner is FH parallel to GI, and therefore FHIG is a Pgr. and also right-angled. It is equilaterall because FG d = HI d = DB e = CA d = FH d = GI. Wherefore FHIG is a f square described about the circle given. *Which was to be done.* b 18. 3. c 28. 1. d 34. 1. e 15 def. 1. f 29. def. 1.

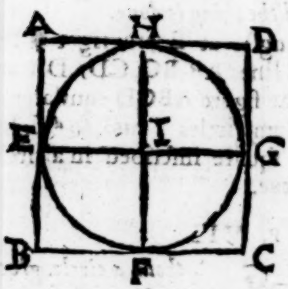
F Schol.

Schol.

A square ABCD described about a circle is double of the square E F G H inscribed in the same circle.

For the rectangle HB = 2 HEF, and HD = 2 HGF by the 41. 1.

Prop. VIII.



In a square given ABCD *to inscribe a circle* IEFGH.

Bisect the sides of the square in the points H, E, F, G cutting one the other in I. a circle drawn from the centre I through H shall be inscribed in the square.

a 7.ax.
b hyp.
c 33. 1.

For because AH and BF are [a] equall and [b] parallel, [c] therefore is AB parallel to HF parallel to DC. After the same manner is AD parallel to EG, parallel to BC; therefore IA, ID, IB, IC are parallelograms. Therefore AH [d] = AE [e] = HI = EI = FI = IG. The circle therefore described from the centre I through H shall passe through H, E, F, G and touch the sides of the square being the angles H, E, F, G are right. *Which was to be done.*

d 7.ax.
e 34. 1.

PROP.

PROP. IX.

About a square given ABCD *to describe a circle* EABCD.

Draw the diameters AC, BD cutting one the other in E. From the centre E through A describe a circle, then I say that circle is described about the square.

For the angles ABD and BAC are a half of right angles, b therefore EA = EB. After the same manner is EA = ED = EC. The circle therefore described from the centre E passes through A, B, C, D the angles of the square given. *Which was to be done.* a 4. cor. 33. 1. b 6. 1.

PROP. X.

To make an Isosceles triangle ABD, *having each angle at the base* B *and* ADB *double to the remaining angle* A.

Take any right line AB, and divide it in C, a so that AB x BC may be equall to ACq. From the centre A through B, describe the circle ABD; and in this circle b apply BD = AC, and join AD; I say ABD is the triangle required. a 11. 2. b 1. 4.

 For

For, draw DC, and through the points C, D, A. c draw a circle. Now because AB x BC = ACq, d it is evident that BD touches the circle ACD which CD cutteth; e therefore is the angle BDC = A, and therefore the angle BDC + CDA f = A + CDA g = BCD. But BDC + CDA = BDA h = CBD. k therefore the angle BCD = CBD, and therefore DC l = DB = m AC, n wherefore the angle CDA = A = BDC. therefore ADB = 2 A = ABD. *Which was to be done.*

c 5. 4
d 37. 3.
e 32. 3.
f 2. ax.
g 32. 1.
h 5. 1.
k 1. ax.
l 6. 1
m constr.
n 5. 1.

This construction is Analytically found out thus; Take the thing for done, and let the right line DC bisect the angle BDA; a therefore DA, DB :: CA, CB. also because the angle CDA b = $\frac{1}{2}$ ADB c = A, c therefore CA = DC. and because the angle DCB h = A + CDA = 2 A, = B, d thence will be DB = DC. f from whence also DB = CA. and so DA (BA.) CA :: CA, CB. g whence BA x CB = CAq.

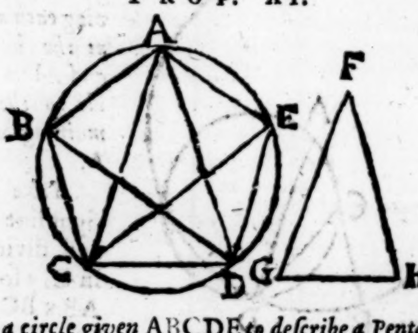
a 3. 6
b constr.
c hyp.
d 6 1.
e 31. 1.
f 2. ax.
g 17. 6

Coroll.

h 32. 1.

Whereas all the angles A, B, D h make up two right angles, it's evident that A is $\frac{1}{5}$ of two right angles.

PROP. XI.



In a circle given ABCDE *to describe a Pentagone figure* ABCDE *equilateral and equiangular.*

a 10. 4
b 2. 4.

a Describe an Isosceles triangle FGH, having each angle at the base double to the other; b inscribe a triangle

triangle CAD equiangular to the ſaid triangle FG-H. [c] Biſect the angles at the baſe ACD & ADC with the right lines DB, CE meeting with the circumference in B and E. join the right lines CB, BA, AE, ED. *Then I ſay it is done.* c 9. 1.

For it is evident by conſtruction that the angles CAD, CDB, BDA, DCE, ECA are equall; wherefore the [d] arches and [e] ſubtended lines DC, CB, BA, AE, DE are equall. Therefore the Pentagone is equilateral, and equiangular [f] becauſe the angles of it BAE, AED, &c. ſtand on equall [g] arches BCDE, ABCD, &c. d 16. 3. e 29. 3. f 27. 3. g 2. ax.

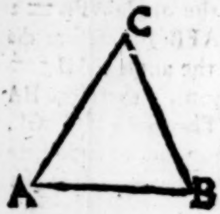
A more eaſy practiſe of this probleme ſhall be deliver'd at 10. 13.

Coroll.

Hence, Each angle of an equilateral and equiangular Pentagone is equall to $\frac{3}{5}$ of two right angles, or $\frac{6}{5}$ of one right angle.

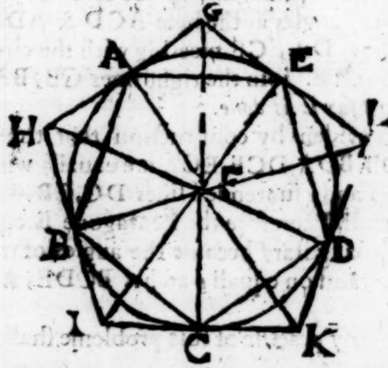
Schol.

Generally all figures of odd number of ſides are inſcribed in circles by the help of Iſoſceles triangles, whoſe angles at the baſe are multiplices of thoſe at the top: and figures of even number of ſides are inſcribed in a circle by the help of Iſoſceles triangles, whoſe angles at the baſe are multiplices ſeſquialter of thoſe at the top. Petr. Herig.



As in the Iſoſceles triangle CAB if the angle A = 3 C = B. then will AB be the ſide of a Heptagone. If A = 4 C; then is AB the ſide of an Enneagone. But if A = $\frac{1}{2}$ C then is A B the ſide of a ſquare. And if A = 2 $\frac{1}{2}$ C A B will ſubtend the ſixt part of a circumference and likewiſe if A = 3 $\frac{1}{2}$ then will AB be the ſide of an Octagone.

PROP. XII.



About a circle given FABCDE, *to describe an equilateral and an equiangular pentagone* HIKLG.

a 11. 4. *a* Inscribe a pentagone ABCDE in the circle given; and from the centre draw the right lines FA, FB, FC, FD, FE; and to those lines draw so many perpendiculars GAH, HBI, ICK, KDL, LEG meeting in the points H, I, K, L, G. then I say it is done.

b cor. 16 13. c 1 cor. 36 3. d 8. 1. For because GA, GE from the same point G *b* touch the circle, *c* therefore is GA = GE, and *d* therefore the angle GFA = GFE, therefore the angle AFE = 2 GFA. After the same manner is the angle AFH = FHB, and consequently the angle AFB = 2 AFH. *e* But the angle AFE = AFB, *f* therefore the angle GFA = AFH. But also the angle FAH *g* = FAG; and the side FA is common, *h* therefore HA = AG = GE = EL, &c. *k* Therefore HG, GL, LK, KI, IH the sides of the pentagone are equall, the angles also are, because double of the equall angles AGF, AHF. therefore, &c.

e 17. 3. f 7. ax. g 12. ax. h 26. 1. k 2. ax.

Coroll.

After the same manner, if any equilateral and equiangled figure be described in a circle, and at the extreme points of the semi-diameters drawn from the centre at angles, be drawn perpendicular lines to the said diameters, I say that these perpendiculars

shall

shall make another figure of as many equall sides and equall angles, described about the circle.

PROP. XIII.

In an equilateral and equiangular pentagone given ABCDE *to inscribe a circle* FGHK.

a Bisect two angles of the pentagone A and B with the right lines AF, BF meeting in the point F. From F draw the perpendiculars FG, FH, FI, FK, FL. Then a circle described from the centre F through G will touch all the sides of the pentagone. a 9.1.

Draw FC, FD, FE. Because BA b = BC and the side BF common, and the angle FBA c = FBC, d therefore is AF = FC and the angle FAB = FCB, but the angle FAB e = $\frac{1}{2}$ BAE = $\frac{1}{2}$ BCD. Therefore the angle FCB = $\frac{1}{2}$ B C D. After the same manner are all the whole angles C, D, E bisected. Now whereas the angle FGB f = FHB, and the ang. FBH = FBG & the side FB is common, g therefore is FG = FH. In like manner are all the right lines FH, FI, FK, FL, FG equal. Therefore a circle described from the centre F through G passes through the points H, I, K, L and h touches the sides of the pentagone because the angles at those points are right. *Which was to be done.*

b hyp. c constr. d 4.1. e hyp. f 11 ax. g 26.1. h cor. 16.3.

Coroll.

Hence, If any two nearest angles of an equilateral and equiangular figure, and from that point in which the lines meet that bisect the angles be drawn right lines to the remaining angles of the figure, all the angles of the figure shall be bisected.

Schol.

By the same method shall a circle be inscribed in any equilateral and equiangular figure.

P R O P. XIV.

About a pentagone given ABCDE *equilateral and equiangular to describe a circle* FABCDE.

Bisect any two angles of the pentagone with the right lines AF, BF meeting in the point F; the circle described from the centre F through A shall be described about the pentagone.

a cor 13. 4. b 6. 1. For let FC, FD, FE be drawn. a Then the angles C, D, E are bisected; b & therefore FA, FB, FC, FD, FE are equall; therefore the circle described from the centre F passes through A, B, C, D, E all the angles of the pentagone. *Which was to be done.*

Schol.

By the same art is a circle described about any figure which is equilateral and equiangular.

PROP.

PROP. XV.

In a circle given GABCDEF *to inscribe an Hexagone (or six-sided figure) equilateral and equiangular* ABCDEF.

Draw the diameter AD; from the centre D through the centre G describe a circle cutting the circle given in the points C and E. Draw the diameters CF, EB; and join AB, BC, CD, DE, EF, FA. *Then I say it's done.*

For the angle CGD [a] = $\frac{1}{3}$ of 2 right ∠ = DGE [b] = AGF [b] = AGB. [c] Therefore BGC = $\frac{1}{3}$ of 2 right = FGE; therefore the [d] arches and [e] subtenses AB, BC, CD, DE, EF are equall. Therefore the Hexagone is equilateral; but it is equiangled also, [f] because all the angles of it stand upon equall arches.

a 32. 1.
b 15. 1.
c cor. 15. 1.
d 26. 3.
e 29. 3.
f 27. 3.

Coroll.

1. Hence, The side of an Hexagone inscribed in a circle is equall to the semidiameter.
2. Hereby an equilateral triangle ACE may very easily be described in a circle given.

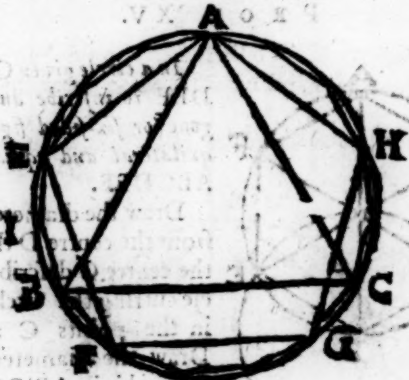
Schol.

To make a true Hexagon upon a right line given CD. Andr. Tacq.

[a] Make an equilateral triangle CGD upon the line given CD; from the centre G through C and D describe a circle. That circle shall contein the Hexagone made upon the given line CD. a 1. 1.

PROP.

Prop. XVI.



In a circle given AEBC *to inscribe a quindecagone (or fifteen-sided figure) equilateral and equiangular.*

a 11.4. b 2.4. — a Inscribe an equilateral pentagone AEFGH in the circle given, and b also an equilateral triangle ABC. then I say BF is the side of the quindecagone required.

c constr. — For the arch AB c is $\frac{1}{3}$ or $\frac{5}{15}$ of that periphery whereof AF is $\frac{2}{5}$ or $\frac{6}{15}$. therefore the remaining part BF is $\frac{1}{15}$ of the periphery; and therefore the quindecagone, whose side is BF, is equilateral; but it is equiangular also d because all the angles insist on equall arches of a circle, whereof every one $\frac{13}{15}$ of the whole circumference. Therefore, &c.

d 27.3.

Schol.

A circle is geometrically divided into parts {
4, 8, 16, &c. by 6, 4, and 9, 1.
3, 6, 12, &c. by 15, 4, and 9, 1.
5, 10, 20, &c. by 11, 4, and 9, 1.
15, 30, 60, &c. by 16, 4, and 9, 1.

Any other way of dividing the circumference into any parts given is as yet unknown; wherefore in the construction of ordinate figures, we are forced to have recourse to mechanick artifices, concerning which you may consult the writers of practical Geometry.

THE

THE FIFTH BOOK OF EUCLIDE'S ELEMENTS.

Definitions.

I. A Part, is a magnitude of a magnitude, a lesse of a greater, when the lesse measureth the greater.

II. Multiplex is a greater magnitude in respect of a lesser, when the lesser measureth the greater.

III. Ratio (or rate) is the mutuall habitude or respect of two magnitudes of the same kind each to other, according to quantity.

In every ratio that quantity which is referred to another quantity is called the antecedent of the ratio, and that to which the other is referrd is called the consequent of the ratio. as in the ratio of 6 to 4, 6 is the antecedent and 4 the consequent.

Note. *The quantity of any ratio is known by dividing the antecedent by the consequent; as the ratio of 12 to 5 is expressed by* $\frac{12}{5}$; *or the quantity of the ratio of* A *to* B *is* $\frac{A}{B}$. *Wherefore often for brevity sake we denote the quantities of rations thus;* $\frac{A}{B} \sqsubset$, or $=$ or $\sqsupset \frac{C}{D}$. *that is, the ratio of* A *to* B *is greater, equall, or lesse then the ratio of* C *to* D. *And this note must be diligently observed in the understanding of the following Book.*

Concerning the diverse species of ratio's, you may please to consult interpreters.

IV. Proportion is a similitude of ratio's.

That which is here termed proportion, *is more rightly called* proportionality *or* analogy; *for proportion*

tion commonly denotes no more then the ratio betwixt two magnitudes.

V. Those numbers are said to have a ratio betwixt them, which being multiplyed may exceed one the other.

E,12.	A, 4. B, 6.	G,24.
F,30.	C,10. D,15.	H,60.

VI. Magnitudes are said to be in the same ratio, the first A to the second B, and the third C to the fourth D, when the equimultiplices E and F of the first A, and the third C compared with the equimultiplices G, H of the second B and the fourth D, according to any multiplication whatsoever, either both together E, F are lesse then GH both together, or equall taken together, or exceed one the other together, if those be taken E, G and F, H, which answer one to the other.

The note hereof is :: ; *as* A.B :: C.D. *That is, as* A *is to* B, *so is* C *to* D. *which signifyes that* A *to* B, *and* C *to* D *are in the same ratio. We sometimes thus expresse it;* $\frac{A}{B} = \frac{C}{D}$. *that is* A.B :: C.D.

VII. Magnitudes that have the same ratio (A.B :: C.D. are called proportional.

E,30.	A, 6. B, 4.	G, 28.
F,60.	C,12, D,9.	H,63.

VIII. When of equimultiplices, E the multiplex of the first magnitude A exceeds the G the multiplex of the second B, but F the multiplex of the third C exceeds not H the multiplex of the fourth D, then the first A to the second B has a greater ratio then the third C to the fourth D.

If $\frac{A}{B} \sqsubset \frac{C}{D}$, *it is not necessary from this definition that* E *should alwayes exceed* G, *when* F *is lesse then* H; *but it is granted that this may be.*

IX. Proportionality consist in three termes at least. *Whereof the second supplyes the place of two.*

X. When 3 magnitudes A, B, C are proportional, the

the firſt A ſhall have a duplicate ratio to the 3 C of that it hath to the ſecond B: But when four magnitudes A, B, C, D are proportional, the firſt A ſhall have a triplicate ratio to the fourth D of what it had to the ſecond B; and ſo alwayes in order one more, as the proportion ſhall be extended.

Duplicate ratio is thus expreſſed $\frac{A}{C} = \frac{A}{B}$ *twice, that is, the ratio of* A *to* C *is double of the ratio of* A *to* B. *Triple ratio is thus expreſſed;* $\frac{A}{D} = \frac{A}{B}$ *thrice. That is, the ratio of* A *to* D *is triple of the ratio of* A *to* B.

∺ *denotes continued proportionals; as* A, B, C, D; *or* 2, 6, 18, 64. *are* ∺

XI. Magnitudes of a like ratio, are antecedents to antecedents, and conſequents to conſequents; *As if* A. B :: C. D. A *and* C; *and* B *and* D *are homologous or magnitudes of a like ratio.*

XII. Alternate proportion is the comparing of antecedent to antecedent, and conſequent to conſequent. *As if* A. B :: C. D. *therefore alternately, or by permutation,* A. C :: B. D. *by the* 16. *of* 5.

In this definition, and the 5 *following, names are given to the ſixe wayes of arguing which are often uſed by Mathematicians: the force of which inferences depends on the propoſitions of this book, which are named in their explications.*

XIII. Inverſe ratio is when the conſequent is taken as the antecedent, and ſo compared to the antecedent as the conſequent; *as* A. B :: C D. *therefore generally* B. A :: D. C. *by cor.* 4. 5.

XIV. Compounded ratio is when the antecedent and conſequent taken both as one are compared to the conſequent it ſelf. *As* A. B :: C. D. *therefore by compoſition* A + B. B :: C + D. D *by* 18. 5.

XV. Divided ratio is when the exceſſe wherein the antecedent exceedeth the conſequent, is compared to the conſequent. *As* A. B :: C. D. *therefore by diviſion* A - B. B :: C - D. D. *by* 17. 5.

XVI.

XVI. Converse ratio is when the antecedent is compared to the excesse wherein the antecedent exceeds the consequent. *As* A.B :: C.D. *therefore by converse ratio.* A. A - B :: C. C - D. *by the coroll. of the* 19. *of the* 5.

XVII. Proportion of equality is where there are taken more magnitudes then two in one order, and also as many magnitudes in another order, comparing two to two being in the same ratio; it cometh to passe that as in the first order of magnitudes, the first is to the last, so in the second order of magnitudes is the first to the last. Or otherwise: it is a comparison of the extremes together, the mean magnitudes being taken away.

XVIII. Ordinate proportionality is, when, as the antecedent is to the consequent, so is the antecedent to the consequent, and as the consequent is to any other, so is the consequent to any other. *As* A.B :: D.E. *also* B.C :: E.F. *it shall be true also* A.C :: D.F *by the* 22. *of the* 5.

XIX. Inordinate proportion is, when three magnitudes being put, and others also, which are equall to these in multitude, as in the first magnitudes the antecedent is to the consequent, so in the second magnitudes is the antecedent to the consequent: and as in the first magnitudes the consequent is to any other, so in the second magnitudes any other thing to the antecedent. *As* A.B :: F.G. *also* B.C :: E.F, *it shall be true in inordinate proportion,* A.C :: E.G. *by the* 23. *of the* 5.

XX. Any number of magnitudes being put; the proportion of the first to the last is compounded out of the proportions of the first to the second, the second to the third, and the third to the fourth, and so forwards till the proportion arise.

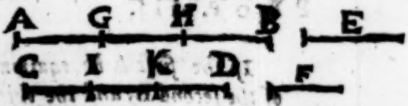
Let there be any number of magnitudes A, B, C, D. by this definition $\frac{A}{D} = \frac{A}{B} + \frac{B}{C} + \frac{C}{D}$.

Axiome.

Axiome.

Magnitudes equimultiplices to the same multiplex, are also equimultiplices betwixt themselves.

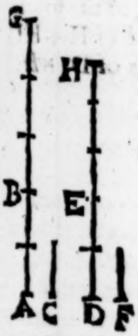
PROP. I.



If there be a number of magnitudes how manysoever, AB, CD *equimultiplices to a like number of magnitudes* E, F *each to other; how multipex one magnitude* AB *is to one* E, *so multiplices are all the magnitudes* AB+CD *to all the other magnitudes* E+F.

Let AG, GH, HB the parts of the quantity AB, be equall to E, and also let CI, IK, KD the parts of the quantity CD be equall to F. The number of these are put equall to those. Now whereas AG+CI a= E+F; a and GH+IK=E+F; a and HB+KD=E+F, it is evident that AB+CD doth so often contain E+F as one AB conteins E. *Which was to be done.* a 1 ax.

PROP. II.

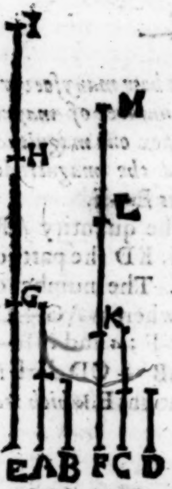


If the first AB *be equimultiplex to the second* C, *as the third* DE *is to the fourth* F, *and if the fifth* BG *be equimultiplex to the second* C *as the sixth* EH *is to the fourth* F; *then shall the first compounded with the fifth* (AG) *be equimultiplex to the second* C, *as the third compounded with the sixth* (DH) *is to the fourth* F.

The number of parts in AB equall each to C is put equall to the numbers of parts in DE, whereof each part is equall to F. Likewise the number of parts in BG is put equall to the number of parts in EH. Therefore the number of parts in AB+BG is equall to the number of parts in DE+EH.

a 2.ax. EH. a That is, the whole line AG is as equimultiplex of C, as the whole line DH is of F. *Which was to be Dem.*

PROP. III.



If the first A *be equimultiplex of the second* B, *and the third* C *of the fourth* D, *and there be taken* EI, FM *equimultiplices of the first and third, then will each of the magnitudes taken be alike equimultiplex of both, the one* EI *to the second* B, *the other* FM *to the fourth* D.

Let EG, GH, HI the parts of the multiplex EI be equall to A, also let FK, KL, LM the parts of the multiplex FM be equall to F,
a hyp. a the number of these is equall to the number of those. Moreover A (that is) EG or GH or HI is put as equimultiplex of B, as C,
b 2.5. or FK &c. of D. b Therefore EG + GH is equimultiplex of the second B, as FK + KL is of the fourth
c 2.5. D. c By the same way of argument is EI (EH+HI) as multiplex of B, as FM (FL+LN) is of D: *Which was to be done.*

PROP. IV.

If the first A *have the same ratio to the second* B , *as the third* C *to the fourth* D; *then also* E *and* F *the equimultiplices of the first* A *and the third* C, *shall have the same ratio to* G *and* H *the equimultiplices of the second* B *and the fourth* D, *according to any multiplication* , *if so taken as they answer each to other* (E. G :: F. H.)

Take I and K the equimultiplices of E and F ; and also L and M the equimultiplices of G & H. a Then is I as multiplex of A , as K of C ; a and also L is as multiplex of B, as M of D. Therefore whereas it is A. B b :: C. D ; according to the sixt definition , if I be ⊏, =, ⊐ L, then consequently after the same manner is K ⊏, =, ⊐ M. Therefore when I and K are taken as multiplices of E and F, as L and M of G, & H, then will it be by the 7 definition E. G :: F. H. *Which was to be Dem.*

a 3. 5.

b hyp.

Coroll.

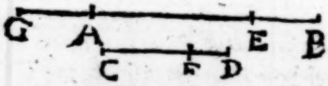
From hence is wont to be demonstrated the proofe of inverse ratio.

For because A. B :: C. D, therefore if E ⊏, =, ⊐ G, then is c likewise F ⊏ = ⊐ H. therefore it is evident that if G ⊏, =, ⊐ E, then is H ⊏, =, ⊐ F; d therefore B. A :: D. C. *Which was to be Dem.*

c 1. def. 5.

d 6. def. 5.

Prop. V.



If a magnitude AB *be as multiplex of a magnitude* CD, *as a part taken from the one* AE *of a part taken from the other* CF; *the residue of the one* EB *shall be as multiplex of the residue of the other* FD *as the whole* AB *is of the whole* CD.

Take any other GA, which shall be as multiplex to FD the residue, as AB is of the whole CD, or as the part taken away AE is of the part taken away C-
a 1. 5. F. a Therefore the whole GA + AE is as multiplex of the whole CF + FD, as the one AE of the one
b 6. ax. C F, that is as A B of C D. therefore G E b = AB;
c 3. ax. and c so A E that was common being taken away, there remains GA = EB.

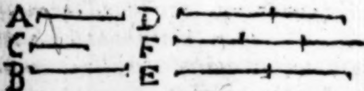
Prop. VI.

If two magnitudes AB, CD *be equimultiplices of two magnitudes* E, F; *and some magnitudes* AG *and* CH *equimultiplices of the same* E, F, *be taken away; then the residues* GB, HD *are either equall to these magnitudes* E, F, *or else equimultiplices of them.*

For because the number of parts in AB, whereof each is equall to E, is put equall to the number of parts in CD, whereof each is equall to F. and also the number of parts in A G equall to the number of parts in C H; If from one you take AG, and from the other CH, a then re-
a 3. ax. mains the number of parts in the remainder GB equall to the number of parts in HD. therefore if GB be once E, then is HD once C. if GB be many times E, then is HD so of C. *Which was to be Dem.*

Prop.

PROP. VII.



Equall magnitudes A *and* B *have to the same magnitude* C *the same proportion or ratio. And one and the same magnitude* C *hath the same ratio to equall magnitudes* A *and* B.

Take D and E equimultiplices of the equall magnitudes A and B, and F any-wise multiplex of C; a 6 *ax.*
then is D [a] = E. Wherefore if D ⊏, =, ⊐ F, b 6 *def.* 5.
then also E will be ⊏, =, ⊐ F. [b] therefore A. C :: c *cor.* 4. 5.
B.C. & [c] by inversion C. A :: [c] C.B. *W.W. to be Dem.*

Schol.

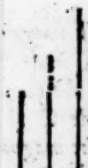
If in stead of the multiplex F, two equimultiplices be taken, it shall be the same way proved that equall magnitudes have the same ratio to other magnitudes that are equall between themselves.

PROP. VIII.

Of unequall magnitudes AB, AC, *the greater* A B *hath a greater ratio to the same third line* D, *then the lesser* AC; *and the same third line* D *hath a greater ratio to the lesser* AC, *then to the greater* AB.

Take EF, EG equimultiplices of the said AB, AC, so that EH being multiplex of D be greater then EG, but lesser then EF. (which will easily happen, if both EG and GF be taken greater then D.) It is manifest from 8 *def.* 5. that $\frac{AB}{D} \sqsubset \frac{AC}{D}$, and $\frac{D}{AB} \sqsupset \frac{D}{AC}$. *Which was to be Dem.*

Prop. IX.



Magnitudes which to one and the same magnitude have the same ratio, are equall the one to the other. And if a magnitude have the same ratio to other magnitudes, those magnitudes are equall one to the other.

1. *Hyp.* If A. C :: B.C; I say that A = B.

A B C For let A be greater or lesse then C. a8.5. [a] then is $\frac{A}{C} \sqsubset$ or $\sqsupset \frac{B}{C}$. *Which is contrary to the Hypothesis.*

2. *Hyp.* If C.B :: C.A. I say that A = B. For let A be $\sqsubset$ B, b8.5. [b] then $\frac{C}{B} \sqsubset \frac{C}{A}$. *Which is against the Hypothesis.*

Prop. X.

Of magnitudes having ratio to the same magnitude, that which has the greater ratio, is the greater magnitude: and that magnitude to which the same carryes a greater ratio, is the lesser magnitude.

A B C 1. *Hyp.* If $\frac{A}{C} \sqsubset \frac{B}{C}$. I say that A $\sqsubset$ B.

a7.5. For if it be said that A = B, [a] then A. C :: B. C. *which is contrary to the Hyp.* If A $\sqsupset$ B, [b] then is

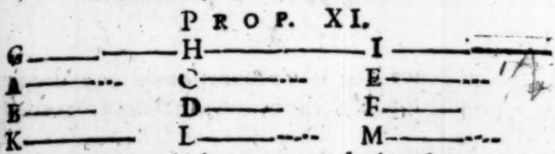
b8.5. $\frac{A}{C} \sqsupset \frac{B}{C}$. *which is also against the Hyp.*

2. *Hyp.* If $\frac{C}{B} \sqsubset \frac{C}{A}$, I say that B $\sqsupset$ A. for if you say B = A, it's against the Hypothesis, for it will

c7.5. [c] follow that C.B :: C.A. If you say B $\sqsubset$ A, [d] then

d8.5. is $\frac{C}{A} \sqsubset \frac{C}{B}$. *Which is also against the Hyp.*

PROP. XI.



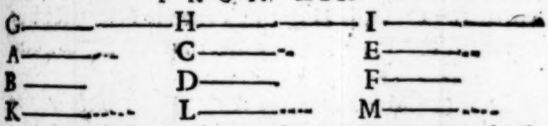
Proportions which are one and the same to any third, are also the same one to another.

Let A. B :: E. F, and C. D :: E. F. I say that A. B :: C. D. Take G, H, I, the equimultiplices of A, C, E; and K, L, M the equimultiplices of B, D, F. Now *a* because A. B :: E. F; if G ⊏, =, ⊐ K, *b* then after the same manner I ⊏, =, ⊐ M. And likewise *a* because E. F :: C. D. if I ⊏, =, ⊐ M, *b* then is H likewise ⊏, =, ⊐ L. *c* Wherefore A. B :: C. D. *Which was to be Dem.* a *hyp.* b 6 *def.* 5. c 6. *def.* 5.

Schol.

Proportions that are one and the same to the same proportions, are the same betwixt themselves.

PROP. XII.



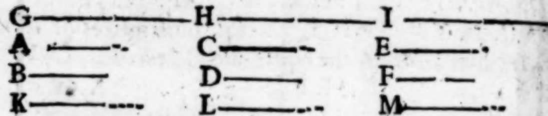
If any number of magnitudes A, B; C, D; E *and* F *be proportionall; as one of the Antecedents* A *is to one of the Consequents* B, *so are all the antecedents* A, C, E *to all the consequents* B, D, F.

Take the equimultiplices of the antecedents G, H, I, and of the consequents K, L, M. Because that as multiplex as one G is of one A, *a* so multiplices are all G, H, I, of all A, C, E; & likewise as multiplex as one K is of one B, so multiplices are all K, L, M, of all B, D, F. moreover because A. B *b* :: C. D *b* :: E. F. if G be ⊏, =, or ⊐ K, then will H likewise be ⊏, =, ⊐ L, and I ⊏, =, ⊐ M. and so if G ⊏, =, ⊐ K in like manner will G + H + I be ⊏, =, ⊐ K + L + M. *c* wherefore A. B :: A + C + E. B + D + F. *Which was to be Dem.* a 1. 5. b *hyp.* c 6 *def.* 5.

Coroll.

From hence, if like proportionals be added to like proportionals, the wholes shall be proportionall.

PROP. XIII.



If the first A *have the same ratio to the second* B, *that the third* C *hath to the fourth* D; *and if the third* C *have a greater proportion to the fourth* D, *then the fift* E *to the sixt* F; *then also shall the first* A *have a greater proportion to the second* B, *then the fift* E *to the sixt* F.

a 6. def. 5.
b 8. def. 5.
c 8. def. 5.

Take G, H, I equimultiplices of A, C, E, and K, L, M, equimultiplices of B, D, F. Now because that A. B :: C. D, if H $\sqsubset$ L, *a* then is G $\sqsubset$ K. but because $\frac{C}{D} \sqsubset \frac{E}{F}$, *b* it may be that H $\sqsubset$ L, and yet I not $\sqsubset$ M. *c* Therefore $\frac{A}{B} \sqsubset \frac{E}{F}$. *Which was to be Dem.*

Schol.

But if $\frac{C}{D} \sqsupset \frac{E}{F}$, then also is $\frac{A}{B} \sqsupset \frac{E}{F}$. Also, if $\frac{A}{B} \sqsubset \frac{C}{D} \sqsubset \frac{E}{F}$, then is $\frac{A}{B} \sqsubset \frac{E}{F}$. And if $\frac{A}{B} \sqsupset \frac{C}{D} \sqsupset \frac{E}{F}$, then is $\frac{A}{B} \sqsupset \frac{E}{F}$.

PROP.

PROP. XIV.

If the first A *have the same ratio to the second* B, *that the third* C *hath to the fourth* D; *and if the first* A *be greater then the third* C; *then shall the second* B *be greater then the fourth* D. *But if the first* A *be equall to the third* C, *then the second* B *shall be equall to the fourth* D. *but if* A *be lesser, then is* B *also lesser.*

Let A $\sqsubset$ C. a then $\frac{A}{B} \sqsubset \frac{C}{B}$. b but $\frac{A}{B} = \frac{C}{D}$. therefore $\frac{C}{D} \sqsubset \frac{C}{B}$. c therefore B $\sqsubset$ D. By the like way of argument, if A $\sqsupset$ C, d then is B $\sqsupset$ D. But if A be put equall to C, then C.B :: e A.B f :: C.D. g therefore B $=$ D. *Which was to be Dem.* a 8.5. b hyp. c 13.5. d 10.5. e 7.5. f hyp. g 11.5.&9.5.

Schol.

By an argument *a fortiori*, if $\frac{A}{B} \sqsupset \frac{C}{D}$, and A $\sqsubset$ C, then is B $\sqsubset$ D. Likewise if A $=$ B, then is C $=$ D. and if A $\sqsubset$, or $\sqsupset$ B, then also is C $\sqsubset$ or $\sqsupset$ D.

PROP. XV.

Parts C *and* F *are in the same ratio, with their like multiplices* AB *and* DE, *if taken correspondently.* (AB. DE :: C.F.)

Let AG, GB parts of the multiplex AB be equall to C; and let DH, HE parts of the multiplex DE be equall to F. a The number of these parts is equall to the number of those. Therefore whereas b AG. C :: DH. F, and GB. C :: HE. F, therefore is c AG + GB (AB.) DH + HE (DE) :: C.F. *Which was to be Dem.* a hyp. b 7.5. c 12.5.

PROP. XVI.

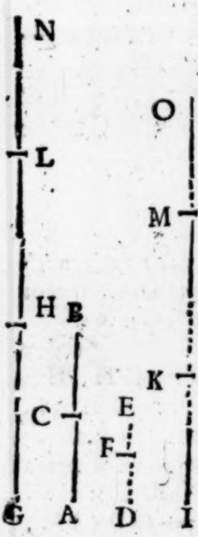
If four magnitudes A, B, C, D *be proportionall, they also shall be alternately proportionall* (A. C :: B. D.)

Take E and F equimultiplices of A and B; take
a 15. 5. also G and H equimultiplices of C and D. Therefore
b hyp. E. F *a* :: A. B *b* :: C. D *a* :: G. H. Wherefore if E ⊏,
c 11. 5. & 14. 5. =, ⊐ G, *c* then likewise is F ⊏, =, ⊐ H.
d 6 def. 5. *d* Therefore A. C :: B. D. *Which was to be Dem.*

Schol.

Alternate ratio has place onely then when the quantities are of the same kind. For heterogeneous quantities are not compared together.

PROP. XVII.

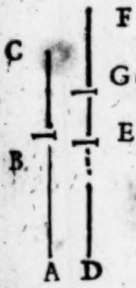


If magnitudes compounded be proportionall (A B. C B :: D E. F E,) *they shall be proportionall also when divided.* (A C. C B :: D F. F E.

Take G H, H L, I K, K M, in order the equimultiplices of A C, C B, D F, F E; and also L N, M O, the equimultiplices of C B, F E. The whole G L is *a* as
a 1. 5. multiplex of the whole A B, as
b constr. one G H of one A C, *b* that is as I K of D F, *c* or as the whole I M
d 1 5. of the whole D E. Also H N (H L + L N) is as *d* multiplex of C B, as K O (K M + M O) is of F E. Therefore, whereas by Hyp. A B. BC :: D E. E F.
e 2. 5. if G L be ⊏, =, ⊐ H N, then likewise *e* will I M ⊏, =, ⊐

KO. Take from these H L, K M that are equall; and if the remainder GH be ⊏, =, ⊐ L N, *f* then will I K ⊏, =, ⊐ M O. *g* whence A C. CB :: D F. F E. *Which was to be Dem.* e 6. *def.* 5. f 5 *ax.* g 6. *def.* 5.

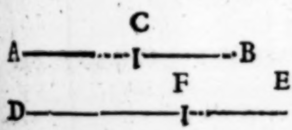
Prop. XVIII.



If magnitudes divided be proportionall (AB.BC :: DE. EF.) *the same also being compounded shall be proportionall* (AC.CB :: DF. FE.)

For if it can be, let AB. CB :: DF. FG ⊐ FE. *a* Then by division will AB. BC :: DG. GF. *b* that is, DG. GF :: DE. EF. and being DG ⊏ DE, *c* therefore is GF ⊏ EF. *d Which is Absurd.* The like absurdity will follow if it be said AB. CB :: DE. GF ⊏ FE. a 17. 5. b *hyp.* & 11. 5. c 14. 5. d 9. *ax.*

Prop. XIX.



If the whole AB *be to the whole* DE *as the part taken away* AC *is to the part taken away* DF; *then shall the residue* CB *be to the residue* FE *as the whole* AB *is to the whole* DE.

Because *a* AB. DE :: AC. DF, *b* therefore by permutation AB. AC :: DE. DF. *c* and thence by division AC. CB :: DF. FE. *b* wherefore again by permutation AC. DF :: CB. FE. *d* that is, AB. DE :: CB, FE. *W. W. to be Dem.* a *hyp.* b 10. 5. c 17. 5. d *hyp.* & 11. 5.

Coroll.

Hence, If like proportionals be substracted from like proportionals, the residues shall be proportionall.

2. *Hence is converse ratio demonstrated.*

Let AB.CB :: DE.FE. I say that AB. AC :: DE. DF. For by *a* permutation AB.DE :: CB.FE, *b* therefore AB. DE :: AC.DF. whence again by permutation AB. AC :: DE. DF. *W. W. to be Dem.* a 16. 5. b 19. 5.

Prop.

Prop. XX.

If there be three magnitudes A, B, C, *and others* D, E, F *equall to those in number, which being tak n two & two in each order are in the same ratio*, (A.B :: D.E; *and* B.C :: E.F,) *and if of equality the first* A *be greater then the third* C; *then shal the fourth* D *be greater then the sixth* F. *But if the first* A *be equall to the third* C, *then the fourth* D *is so to the sixth* F; & *if* A *be lesse then* C, *so* D *is lesse then* F.

a hyp. b cor 4. 5. c hyp & 8. 5. d schol. 13. 5.

1. *Hyp.* Let A $\sqsubset$ C. Because [a] E.F :: B.C. by [b] inversion shall be F.E :: C.B. [c] But $\frac{C}{B} \sqsupset \frac{A}{B}$, [d] therefore $\frac{F}{E} \sqsupset \frac{A}{B}$ or $\frac{D}{E}$, [e] therefore D $\sqsubset$ F. *W.W. to be Dem.*

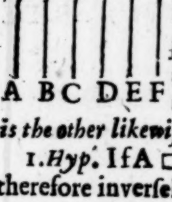
e 10. 5.

2. *Hyp.* By the same way of argument, if A $\sqsupset$ C, it will appear that D $\sqsupset$ F.

f 7. 5. g 11. 5. and 9. 5.

3. *Hyp.* If A $=$ C. Because F. E :: C.B :: [f] A.B :: D.E. [g] therefore is D $=$ F. *W.W. to be Dem.*

Prop. XXI.



If there be three magnitudes A, B, C, *and others also* D, E, F *equall to them in number, which taken two and two are in the same ratio; and their proportion inordinate* (A. B :: E. F. *and* B.C :: D.E.) & *if of equality, the first* A *be greater then the third* C; *then is the fourth* D *greater then the sixth* F: *but if the first be equall to the third, then is the fourth equall to the sixth; if lesse, so is the other likewise.*

a hyp. b 8. 5. c schol. 13. 5. d 10. 5.

1. *Hyp.* If A $\sqsubset$ C; then because [a] D. E :: B. C, therefore inversely E.D :: C.B. but $\frac{C}{B}$ [b] $\sqsupset \frac{A}{B}$; [c] therefore $\frac{E}{D} \sqsupset \frac{A}{B}$, that is $\frac{E}{F}$ [d] therefore D $\sqsubset$ F,

2. *Hyp.*

2.*Hyp*. By the like argument, if A ⊐ C, then is D ⊐ F.

3.*Hyp*. If A = C; then becau se E. D :: *e* C.B :: A.B :: *f* E F, *g* therefore is D = F. *W.W. to be Dem.* e 7.5. f hyp. g 9.5.

Prop. XXII.

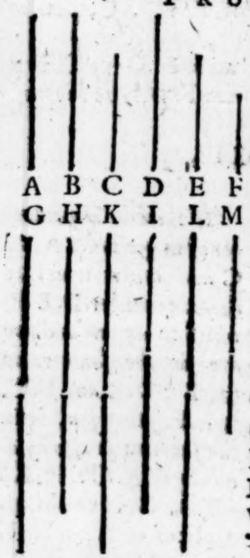
If there be any number of magnitudes A, B, C, *and others equall to them in number* D, E, F, *which taken two and two are in the same ratio* (A.B :: D.E. *and* B.C. :: E.F.) *they shall be in the same ratio also by equality*, (A.C :: D.F.)

Take G, H equimultiplices of A, D; & I, K of B, E; and also L, M of E. F.

Because *a* A.B :: D. E, *b* therefore G. I :: H.K. and in like manner I.L :: K.M. therefore, if G ⊏, =, ⊐ L, *c* then is H ⊏, =, ⊐ M, *d* therefore A.C :: D.F. By the same way of demonstration if further C.N :: F.O, then by equality A.N :: D.O. *W.W. to be Dem.* a hyp. b 4.5. c 20.5. d 6. def. 5.

Prop.

Prop. XXIII.



If there be three magnitudes A, B, C, and others D, E, F, equall to them in number, which taken two and two are in the same ratio, and their proportionality inordinate (A B :: E. F. and B.C :: D. E.) they shall be in the same ratio also by equality.

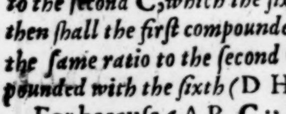
Take G, H, I equimultiplices of A, B, D; and also K, L, M equimultiplices of C, E, F.
a 15. 5. b hyp. Then G. H :: a A. B :: b E. F a :: L.M. Moreover because b B. C :: D.E, thence is a H.K :: I. L; therefore G, H, K, and I,
c 4. 5. L, M are according to 21. 5. Wherefore if G be $\sqsubset$, $=$, $\sqsupset$ K, then is likewise I $\sqsubset$,
d 6. def. 5. $=$, $\sqsupset$ M. and so d consequently A. C :: D. F. *W. W. to be Dem.*

If there be more magnitudes then three, this way of demonstration holds good in them also.

Coroll.

* 22. and 23. 5. & 20. def. 1. From hence * it follow's that ratio's compounded of the same ratio's, are among themselves the same; as also that the same parts of the same ratio's, are among themselves the same.

Prop. XXIV.

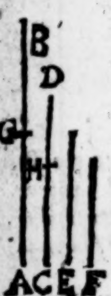


If the first magnitude A B have the same ratio to the second C, which the third D E hath to the fourth F; & if the fifth BG have the same ratio to the second C, which the sixth E H hath to the fourth F, then shall the first compounded with the fifth (AG) have the same ratio to the second C, which the third compounded with the sixth (D H) hath to the fourth F.

a hyp. For because a A B. C :: D E. F, and by the Hyp. and

and inversion C. BG :: F. EH; therefore by *b* equality AB. BG :: DE. EH. whence by compounding, AG. BG :: DH. EH. Also *c* BG. C :: EH. F. Therefore again by *b* equality A G. C :: D H. F. *W.W.to be Dem.* b 22. 5. c hyp.

P R O P. XXV.



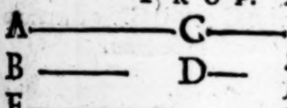
If four magnitudes be proportionall (AB. CD :: E.F) *the greatest* AB *and the least* F *shall be greater then the rest* C D, *and* E.

Make AG = E, and CH = F. Because AB. CD :: *a* E. F :: *b* AG. CH, *c* thence is AB.CD :: GB.HD. *d* but AB ⊏ CD. *e* therefore GB ⊏ HD. But AG + F = E + CH, therefore AG + F + GB ⊏ E + CH + HD, that is, AB + F ⊏ E + CD. *W.W. to be Dem.*

a hyp. b 7. 5. c 19. 5. d hyp. e schol. 14. 5.

These propositions which follow are not Euclid's, *but taken out of other Authors, and here subjoyned because of their frequent use.*

P R O P. XXVI.

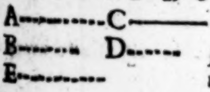


If the first have a greater proportion to the second; then the third to the fourth, then contrarywise, by conversion, the second shall have a lesse proportion to the first, then the fourth to the third.

Let $\frac{A}{B} \sqsubset \frac{C}{D}$. I say that $\frac{B}{A} \sqsupset \frac{D}{C}$. For conceive $\frac{C}{D} = \frac{E}{B}$; *a* therefore $\frac{A}{B} \sqsubset \frac{E}{B}$. *b* whence A ⊏ E. *c* therefore $\frac{B}{A} \sqsupset \frac{B}{E}$, *d* or $\frac{D}{C}$. *W.W. to be Dem.*

a 13. 5. b 10. 5. c 8 5. d cor. 4 5.

P R O P. XXVII.



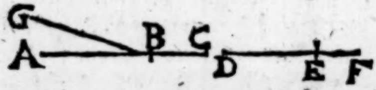
If the first have a greater proportion to the second, then the third to the fourth; then alternatly the first shall have a greater proportion to the third, then the second to the fourth.

Let

Let $\frac{A}{B} \sqsubset \frac{C}{D}$. then I say $\frac{A}{C} \sqsubset \frac{B}{D}$. For conceive

a 10.5. b 1.5. c 16.5. $\frac{E}{B} = \frac{C}{D}$. a therefore $A \sqsubset E$, b and therefore $\frac{A}{C} \sqsubset \frac{E}{C}$ c or $\frac{B}{D}$. *W.W. to be Dem.*

Prop. XXVIII.

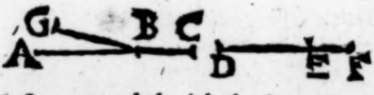


If the first have a greater proportion to the second then the third to the fourth, then the first compounded with the second shall have a greater proportion to the second, then the third compounded with the fourth to the fourth.

Let $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$. I say that $\frac{AC}{BC} \sqsubset \frac{DE}{EF}$. For con-

a 10.5. b 4 ax. c 8.5. d 18.5. ceive $\frac{GB}{BC} = \frac{DE}{EF}$. a therefore is $AB \sqsubset GB$. adde BC to each part, then b will $AC \sqsubset GC$. c therefore $\frac{AC}{BC} \sqsubset \frac{GC}{BC}$. d that is $\frac{DF}{FE}$. *W.W. to be Dem.*

Prop. XXIX.



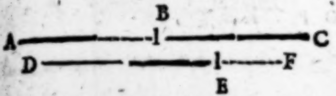
If the first compounded with the second have a greater proportion to the second, then the third compounded with the fourth hath to the fourth; then by division the first shall have a greater proportion to the second, then the third to the fourth.

Let $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$. then I say $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$. For con-

a 10.5. ceive $\frac{GC}{BC} = \frac{DE}{EF}$. a therefore $AC \sqsubset GC$. Take away

b 5 ax. c 8.4. d 17.5. BC, that is common; then b remains $AB \sqsubset GB$. c Therefore $\frac{AB}{BC} \sqsubset \frac{GB}{BC}$ d or $\frac{DE}{EF}$. *W.W. to be Dem.*

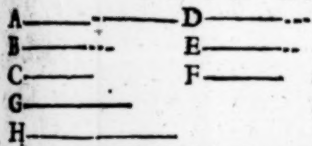
PROP. XXX.



If the first compounded with the second have a greater proportion to the second, then the third compounded with the fourth hath to the fourth, then by converse ratio shal the first compounded with the second have a lesser ratio to the first, then the third compounded with the fourth shall have to the third.

Let $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$. Then I say that $\frac{AC}{AB} \sqsupset \frac{DF}{DE}$. For because that $\frac{AC}{BC}$ *a* $\sqsubset \frac{DF}{EF}$ *b* therefore by division $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$ by conversion *c* therefore $\frac{BC}{AB} \sqsupset \frac{EF}{DE}$ and *d* therefore by composition $\frac{AC}{AB} \sqsupset \frac{DF}{DE}$. *W. W. to be Dem.* a hyp. b 29. 5. c 26. 5. d 18. 5.

PROP. XXXI.

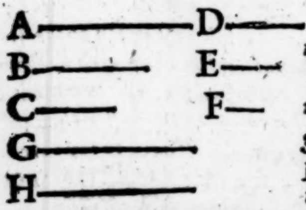


If there be three magnitudes A, B, C, & others also D, E, F equall to them in number; & if there be a greater proportion of the first of the former to the second, then there is of the first of the last to their second $\left(\frac{A}{B} \sqsubset \frac{D}{E};\right)$ *and and there be also a greater proportion of the second of the first magnitudes to the third, then there is of the second of the last magnitudes to their third* $\left(\frac{B}{C} \sqsubset \frac{E}{F}.\right)$ *Then by equality also shall the ratio of the first of the former magnitudes to the third be greater then the ratio of the first of the latter magnitudes to the third* $\left(\frac{A}{C} \sqsubset \frac{D}{F}.\right)$

Conceive $\frac{G}{C} = \frac{E}{F}$. *a* therefore is B $\sqsubset$ G, & *b* therefore $\frac{A}{G} \sqsubset \frac{A}{B}$. Again conceive $\frac{H}{G} = \frac{D}{F}$. *c* therefore $\frac{H}{G} \sqsupset \frac{A}{B}$, therefore much more $\frac{H}{G} \sqsupset \frac{A}{G}$. *d* wherefore A $\sqsubset$ H, *e* & consequently $\frac{A}{C} \sqsubset \frac{H}{C}$ *f* or $\frac{D}{F}$. a 10. 5. b 8. 5. c 13. 5. d 10. 5. e 8. 5. f 22. 5.

PROP.

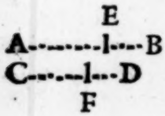
Prop. XXXII.



If there be three magnitudes A, B, C, *and others* D, E, F, *equall to them in number; and there be a greater proportion of the first of the former magnitudes to the second, then there is of the second of the latter to the third* $\left(\frac{A}{B} \sqsubset \frac{E}{F}\right)$ *and also the ratio of the second of the former to the third be greater then the ratio of the first of the latter to the second* $\left(\frac{B}{C} \sqsubset \frac{D}{E}\right)$ *then by equality also shall the proportion of the first of the former to the third, be greater then that of the first of the latter to the third* $\left(\frac{A}{C} \sqsubset \frac{D}{E}\right)$

The demonstration of this proposition is altogether like that of the precedent.

Prop. XXXIII.

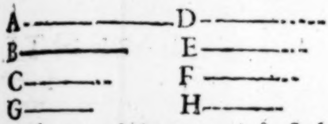


If the proportion of the whole AB *to the whole* CD *be greater then the proportion of the part taken away* AE *to the part taken away* CF; *then shall also the ratio of the remainder* EB *to the remainder* FD *be greater then that of the whole* AB *to the whole* CD.

a hyp.
b 27.5.
c 30.5.

Because that $\frac{AB}{CD}$ a $\sqsubset \frac{AE}{EF}$, b therefore by permutation $\frac{AB}{AE} \sqsubset \frac{CD}{CF}$, c therefore by converse ratio $\frac{AB}{EB} \sqsupset \frac{CD}{FD}$, and by permutation again $\frac{AB}{CD} \sqsupset \frac{EB}{FD}$. *W. W. to be Dem.*

Prop. XXXIV.



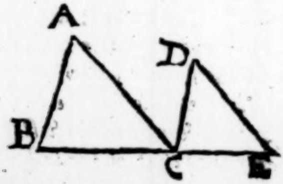
If there be any number of magnitudes, and others also equall to them in number; and the proportion of the first of the former to the first of the latter be greater then that of the second to the second, and that greater then the proportion of the third to the third, and so forward: all the former magnitudes together shall have a greater ratio to all the latter together, then all the former, leaving out the first, shall have to all the latter, leaving out the first; but lesse then that of the first of the former to the first of the latter; and lastly greater then that of the last of the former to the last of the latter.

You may please to consult Interpreters for the demonstration hereof, we having for brevities sake omitted it, and because 'tis of no use in these Elements.

The End of the fifth Book.

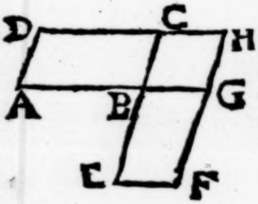
THE SIXTH BOOK OF EUCLIDE'S ELEMENTS.

Definitions.

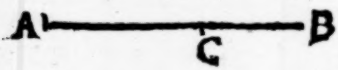


I. Like right-lined figures (ABC, DCE) are such whose severall angles are equall one to the other, and also their sides about the equall angles, proportional.

The angle B = DCE, *and* AB. BC :: DC. CE. *Also the angle* A = D, *and* BA. AC :: CD. DE. *Lastly the angle* ACB = E, *and* BC. CA :: CE. ED.

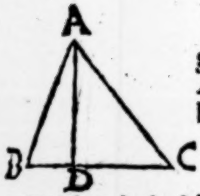


II. Reciprocal figures are (BD, BF) when in either figure are the terms antecedents and consequents of ratio's (that is, AB. BG :: EB. BC.)



III. A right line AB is said to be cut according to mean and extreme proportion, when as the whole AB is to the greater segment AC, so is the greater segment AC to the lesse CB (AB. AC :: AC. CB.)

IV.



I V. The altitude of any figure ABC, is a perpendicular line AD drawn from the top A to the base BC.

V. A ratio is said to be compounded of two ratio's, when the quantities of the ratio's being multiplyed the one into the other, do produce any ratio. *As the ratio of* A *to* C *is compounded of the ratio's of* A *to* B *and* B *to* C. *For* $\frac{A}{B} + \frac{B}{C} \,{}^{a}\!= \frac{A}{C} \,{}^{b}\!= \frac{AB}{BC}$. a 10. *def.* 5. b 15. 4.

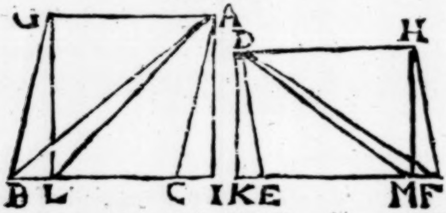
PROP. I.

Triangles ABC, ACD, *and parallelograms* BCAE, CDFA, *which have the same height, are in proportion one to the other, as their bases,* BC, CD *are.*

a Take as many as you please, BG, GH, equall to BC, and also DI $=$ CD. and join AG, AH, AI. a 3. 1.

b The triangles ACB, ABG, AGH are equall, and b also the triangle ACD $=$ ADI. Therefore the triangle ACH is as multiplex of the triangle ACB, as the base HC is of the base BC; and the triangle ACI as multiplex of the triangle ACD, as the base CI is of CD. But if HC $\sqsubset$, $=$, $\sqsupset$ CI, c then is likewise the triangle AHC $\sqsubset$, $=$, $\sqsupset$ ACI; and d therefore BC. CD :: the triangle ABC. ACD :: e Pgr. CE. CF. *W. W. to be Dem.* b 38. 1. c sch. 38. 1. d 6 *def.* 5. e 41. 1. and 15. 5.

Scholium.



Hence, *Triangles* ABC, DEF *and Pgrs.* AGBC, DEFH, *whose bases* BC, EF *are equall, are in such proportion as their altitudes,* AI, DK *are.*

a 3. 1. *a* Take IL = CB, and KM = EF; and join LA,
b 7. 5. LG, MD, MH. then is it evident that the triang.
c 1. 6 ABC. DEF :: *b* ALI. DKM :: *c* AI. DK :: *d* pgr.
d 41. 1. and 15. 5. AGBC. DEFH. *W. W. to be Dem.*

Prop. II.

If to one side BC *of a triangle* ABC *be drawn a parallel right line* DE, *the same shall cut the sides of the triangle proportionally* (AD. BD :: AE. EC.) *And if the sides of the triangle be proportionally cut* (AD. BD :: AE. EC) *then a right line* DE *joined at the sections* D, E, *shall be parallel to the remaining side of the triangle* BC. Draw CD and BE.

a 37. 1. 1. *Hyp.* Because the Triangle DEB *a* = DEC,
b 7. 5. *b* therefore shall be the triangle ADE. DBE :: ADE.
c 1. 6. ECD. But the triang. ADE. DBE :: *c* AD. DB, and
d 11. 5. the triangle ADE. DEC : : AE. EC, *d* therefore AD DB :: AE. EC.

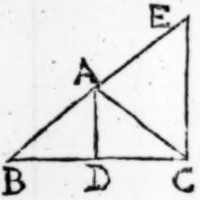
e 1. 6. 2. *Hyp.* Because AD DB :: AE. EC. *e* that is the tri-
f 9. 5. angle ADE. DBE :: ADE. ECD; *f* therefore is the
g 39. 1 triangle DBE = ECD; and *g* therefore DE, BC are parallels. *W. W. to be Dem.*

Scholium.

If there be drawne many parallelles to one side of any triangle, then all the segments of the sides shall

ſhall be proportionall; as is eaſily deducible from the precedent.

Prop. III.

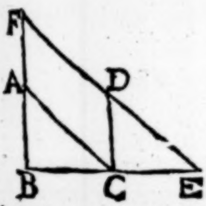


If an angle BAC *of a triangle* BAC *be biſected, and the right line* AD *that biſects the angle, cut the baſe alſo; then ſhall the ſegments of the baſe have the ſame ratio that the other ſides of the triangle have,* (BD. DC :: AB. AC.) *And if the ſegments of the baſe have the ſame ratio, that the other ſides of the triangle have* (BD. DC :: AB. AC) *then a right line* AD *drawn from the top* A *to the ſection* D, *ſhall biſect that angle* BAC *of the triangle.*

Produce BA, and make AE = AC, and join CE.

1.*Hyp.* Becauſe AE = AC, therefore is the angle a 5. 1.
ACE *a* = E *b* = $\frac{1}{2}$ BAC *c* = DAC; *d* therefore b 32. 1. c *hyp.*
DA, CE are parallels. *e* Wherefore BA. AE d 27. 1.
(AC) :: BD. DC. e 2. 6.
2.*Hyp.* Becauſe BA.AC (AE) :: BD.DC, *f* there- f 2. 6.
fore are DA, CE parallels; and *g* therefore is the an- g 29. 1. h 5. 1.
gle BAD = E; and the angle DAC *g* = ACE *h* = k 1. ax.
E. *k* therefore the angle BAD = DAC. Wherefore the angle BAC is biſected. *W.W. to be Dem.*

Prop. IV.



Of equiangular triangles ABC, DCE, *the ſides are proportionall which are about the equall angles,* B, DCE, (AB. BC :: DC. CE, &c.) *and the ſides* AB, DC, *&* C *which are ſubtended under the equall angles* ACB, E, &c. *are homologous, or of like ratio.*

Set the ſide BC in a direct line to the ſide CE, and produce BA and ED till they *a* meet. a 32. 1.

b *hyp.* c 28. 1. Because the angle B b = ECD, c therefore BF, CD are parallel: Also because the angle BCA b = CED, c therefore are CA, EF parallel. Therefore d 34. 1. the figure CAFD is a Pgr. d therefore AF = CD, and AC = d FD. Whence it is evident that AB. AF e 2. 6. f 16 5. (CD) :: e BC. CE. f by permutation therefore AB. BC :: CD. CE. also BC. CE :: FD (AC.) DE. f and thence by permutation BC. AC :: CE. DE. g 22. 5. g Wherefore also by equality AB. AC :: CD. DE. Therefore, &c.

Coroll:

Hence AB. DC :: BC. CE :: AC. DE.

Schol.

Hence, If in a triangle FBE there be drawn AC a parallel to one side FE, the triangle ABC shall be like to the whole FBE.

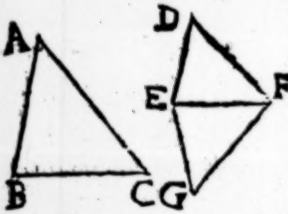
PROP. V.

If two triangles ABC, DEF *have their sides proportionall* (AB. BC :: DE. EF, *and* AC. BC :: DF. EF, *& also* AB. AC :: DE. DF) *those triangles are equiangular, and those angles equall under which are subtended the homologous sides.*

At the side EF a make the angle FEG = B, and a 23 1. b 32. 1. the angle EFG = C; b whence the angle G = A. c 4. 6. d *hyp.* Therefore GE. EF c :: AB. BC :: d DE. EF. e and therefore GE = DE. Likewise GF. FE c :: AC. CB :: d DF. FE. e therefore GF = DF. Therefore the e 11. 5. and 9. 5. f 8. 1. triangles DEF, GEF are mutually equilaterall. f Therefore the angle D = G = A, and the angle FED f = FEG = B, and g consequently the angle g 32. 1. DFE = C. Therefore, &c.

PROP.

PROP. VI.

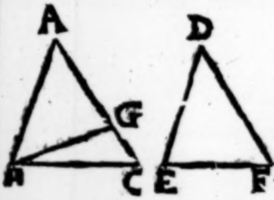


If two triangles ABC, DEF *have one angle* B *equall to one angle* DEF, *and the sides about the equal angles* B, DEF *proportionall* (AB. BC :: DE. EF) *then those triangles* ABC, DEF *are equiangular, and have those angles equall, under which are subtended the homologous sides.*

At the side EF make the angle FEG = B, and the angle EFG = C; *a* then will the angle G = A. Therefore GE. EF :: *b* AB. BC :: *c* DE. EF, *d* and therefore DE = GE. But the angle DEF *e* = B *f* = GEF; therefore the angle D *g* = G = A, *h* and consequently the angle EFD = C. *W. W. to be Dem.*

a 32. 1.
b 4. 6.
c hyp.
d 9. 5.
e hyp.
f constr.
g 4. 1.
h 32. 1.

PROP. VII.

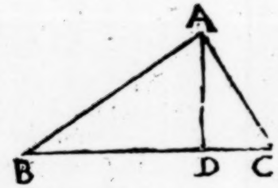


If two triangles ABC, DEF *have one angle* A *equall to one angle* D, *and the sides about the other angles* ABC, E, *proportional* (AB. BC :: DE. EF) *and if they have both of the remaining angles* C, F *either lesse or not lesse then a right-angle; then shall the triangles* ABC, DEF *be equiangular, and have those angles equall about which the proportionall sides are.*

For, if it can be, let the angle ABC $\sqsubset$ E, and make the angle ABG = E. Therefore, whereas the angle A *a* = D, *b* thence is the angle AGB = F. Therefore AB. BG *c* :: DE. EF :: *d* AB. BC. *e* therefore BG = BC. *f* therefore the angle BGC = BCG. *g* Therefore BGC or C is lesse then a right angle, & *h* consequently AGB or F is greater then a right: Therefore the angles C and F are not of the same species or kind, which is against the Hypothesis.

a hyp.
b 32. 1.
c 4. 6.
d hyp.
e 9. 5.
f 5. 1.
g cor. 17. 1.
h cor. 13. 1.

PROP. VIII.



If in a right-angled triangle ABC, *from the right-angle* BAC *there be drawn* AD *a perpendicular to the base* BC; *then the triangles about the perpendicular* (ADB, ADC) *are like both to the whole triangle* ABC, *and also one to the other.*

a hyp. b 12. ax. c 32. & 4. 6. d 21. 6.

For because BAC, ADB are a right angles, b and so equall, and B common; the triangles BAC, ADB c are like. By the same argum. BAC, ADC are like d whence also ADB, ADC will be like. *W. W. to be Dem.*

Coroll.

e 1. def. 6.

Hence, 1. BD. DA e :: DA. DC.

2. BC. AC :: AC. DC. and CB. BA :: BA. BD.

PROP. IX.

From a right line given AB *to cut off any part required, as* $\frac{1}{3}$ (AG.)

a 3. 1. b 31. 1.

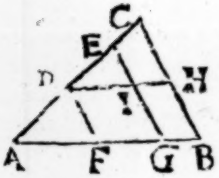
From the point A *draw an infinite line* AC *any-wise, in which* a *take any three equall parts* AD, DE, EF. *join* FB, *to which from* D b *draw the parallel* DG; *and the thing is done.*

c 2. 6. d 18. 5.

For GB. AG :: c FD. AD; whence by d composition AB. AG :: AF. AD. therefore, whereas AD = $\frac{1}{3}$ of AF, therefore is AG = $\frac{1}{3}$ of AB. *W. W. to be Done.*

PROP.

PROP. X.

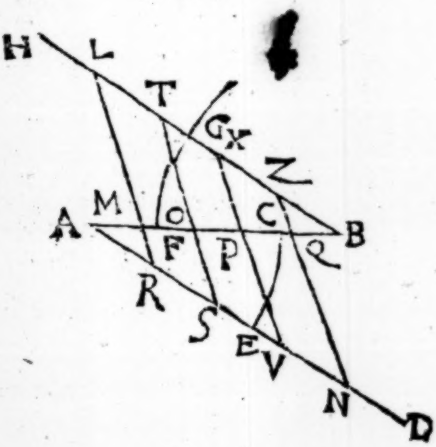


To divide a right line given AB *not divided* (in F and G) *as another line given* AC *was cut* (in D and E.)

Let a right line BC join the extremities of the line divided, and of the line not divided; and to that line from the points E,D *a* draw the parallels E G, D F meeting with the right line that is to be cut, in G & F; Then the thing is Done. a 31. 1.

For let DH be *a* drawn parallel to AB. Then AB. DE :: *b* AF. FG. and DE. EC *b* :: DI. IH :: *c* FG. GB. *W. W. to be Done.* b 2. 6 c 34 1. *and* 7. 5.

Schol.



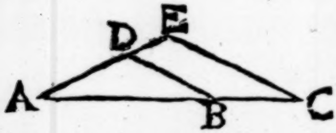
Hence is learnt to cut a right line given AB *into as many equall parts as you please* (*suppose* 5;) which will be more easily performed thus.

Draw an infinite line AD, and another BH parallel to it and infinite also. Of those take equall parts, AR, RS, SV, VN; and BZ, ZX, XT, TL; in

in each line lesse parts by one, then are required in AB; then let the right lines LR, TS, XV, ZN be drawn; these lines so drawn shall cut the right line given AB into five parts.

a 31. 1. b constr. c 2.6. For RL, ST, VX, NZ are *a* parallels; therefore, whereas AR, RS, SV, VN are *b* equall; *c* thence AM, MO, OP, PQ are equall also. Likewise, because that BZ = ZX, therefore is BQ = PQ and therefore AB is cut into five parts. *W. W. to be Done.*

PROP. XI.



Two right lines being given, AB, AD, *to find out a third in proportion to them* (DE.)

Join BD, and from AB being produced take BC = AD. Through C draw CE parallel to BD; with which let AD produced meet in E. then is DE the proportionall required.

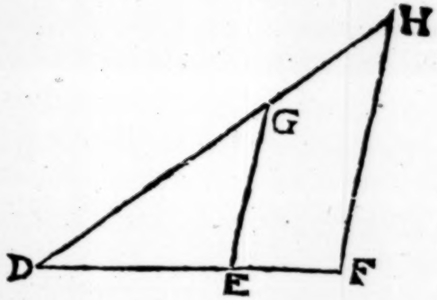
a 2.6. For AB. BC (AD) *a* :: AD. DE. *W. W. to be Done.*

Or thus: make the angle ABC right, and also the angle ACD right.

b cor.9. then *b* AB. BC :: BC. BD.

PROP.

PROP. XII.



Three right lines being given DE, EF, DG, *to find out a fourth proportionall* GH.

Join EG, and thorough F draw FH parallel to EG; with which let DG produced to H meet. Then it is evident that DE. EF *a* :: DG. GH. *W. W. to be Done.* a 2. 6.

PROP. XIII.

Two right lines being given AE, EB, *to find out a mean proportionall* EF.

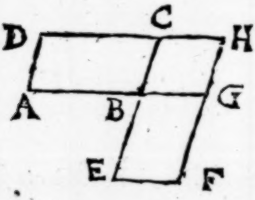
Upon the whole line AB as a diameter describe a semicircle AFB, & from E erect a perpendicular EF meeting with the periphery in F. then AE. EF :: EF. EB. For let AF & FB be drawn; *a* then from the right angle of the right-angled triangle AFB is drawn a right line FE perpendicular to the base. *b* Therefore AE. FE :: FE. EB. *W. W. to be Done.* a 31. 3. b cor. 8. 6.

Co-

Coroll.

Hence, A right line drawn in a circle from any point of the diameter perpendicularly, and extended to the circumference, is a mean proportion all betwixt the two segments of the diameter.

P R O P. XIV.



Equall Parallelograms having one angle ABC *equall to one* EBG, *have the sides* BD, BF *which are about the equall angles reiprocall* (A B. B G :: EB. BC;) *and those Parallelograms* B D, B F *which have one angle* ABC *equall to one* EBG, *and the sides which are about the equall angles reciprocall, are equall.*

For let the sides AB, BG about the equall angles make one right line; a wherefore EB, BC shall doe the same. Let F G, D C be produced till they meet.

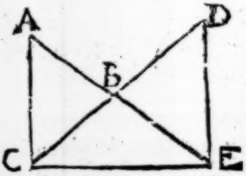
a *sch*.15.1.
b 1. 6.
c 7. 5.
d 1. 6.
e 11. 5.
f 1. 6.
g *hyp*.
h 1. 6.
k 11, & 9. 5.

1.*Hyp*. AB.BG b :: BD. BH :: c BF. BH :: d BE. BC. e therefore, &c.

2.*Hyp*. BD. BH :: f AB. BG :: g BE. BC :: h BF. BH. k Therefore the Pgr. BD = BF. *Which was to be Dem.*

P R O P.

PROP. XV.



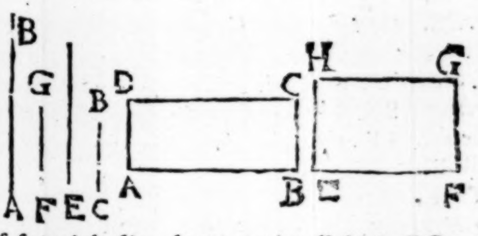
Equall triangles having one angle ABC *equall to one* DBE, *their sides which are about the equall angles are reciprocall* (AB. BE :: DB. BC.) *And those triangles that have one angle* ABC *equall to one* DBE, *and have also the sides that are about the equall angles reciprocall* (AB. BE :: DB. BC.) *are equall.*

Let the sides CB, BC, which are about the equall angles be set in a strait line; a therefore ABE is a right line. Let CE be drawn. a schol. 15. 1.

1. *Hyp.* AB. BE :: b the triangle ABC. CBE c :: the triangle DBE. CBE :: d DB. BC. e therefore, &c. b 1. 6. c 7. 5.

2. *Hyp.* The triangle ABC. CBE :: f AB. BE :: g DB. BC h :: the triangle DBC. CBE. k Therefore the triangle ABC = DBE. *W.W. to be Dem.* d 1. 6. e 11. 5. f 1. 6. g hyp. h 1. 6. k 11, & 9. 5.

PROP. XVI.



If four right lines be proportionall (AB. FG :: EF. CB) *the rectangle* AC *comprehended under the extremes* AD, CB, *is equall to the rectangle* EG *comprehended under the means* FG, EF. *And if the rectangle* AC *comprehended under the extremes* AB. CB *be equall to the rectangle* EG *comprehended under the means* FG, EF, *then are the four right lines proportionall.* (AB. FG :: EF, CB)

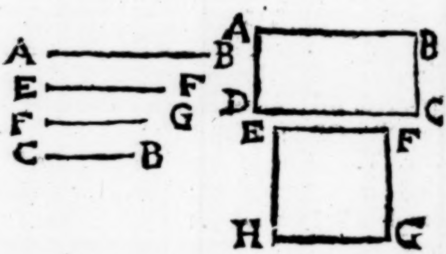
1. *Hyp.*

b 12.ax. 1. *Hyp.* The angles B and F are right, and *a* consequently equall, & by hypothesis AB. FG :: EF. CB.
b 14.6. *b* therefore the rectangle AC = EG.
c hyp. 2. *Hyp.* The rectangle AC *c* = EG, and the angle
d 14.6. B=F; *d* therefore AB. FG :: EF. CB. *W. W. to be Dem.*

Coroll.

e 12.6. Hence, it is easy to apply a rectangle given EG to a right line given AB; (*viz.*) by making AB. EF :: FG. BC.

PROP. XVII.



If three right lines be proportionall (AB. EF:: EF.CB.) *the rectangle* AC *made under the extreames* AB, CB *is equall to the square* EG *made of the middle* EF. *And if the rectangle* AC *comprehended under the extremes* AB, CB *be equal to the square* EG *made of the middle* E F, *then the three lines are proportional,* (AB: EF:: EF. CB).

Take FG=EF.

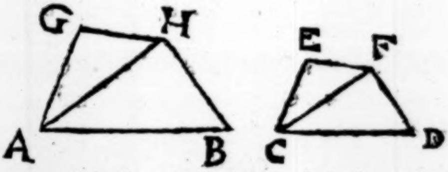
a hyp. 1. *Hyp.* AB. EF :: *a* EF (FG.) CB. therefore the
b 16.6. rectangle AC *b* =EG *c* = EFq.
c 29.def.1.
d hyp. 2 *Hyp.* The rectangle AC *d* = to the square E-
e 16.6. G=EFq. *e* therefore AB. EF :: FG (EF.) BC. *W.W. to be Dem.*

Coroll.

Let A×B = Cq. therefore A.C :: C, B.

PROP.

PROP. XVIII.

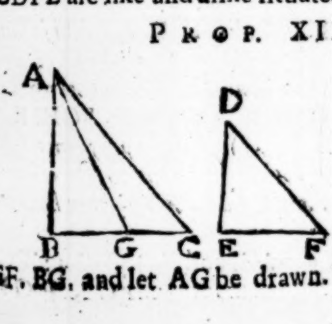


From a right line given AB *to describe a right-lined figure* AGHB *like and alike situate to a right-lined figure given* CEFD.

Resolve the right-lined figure given into triangles; *a* Make the angle ABH=D, *a* & the angle BAH = DCF, *a* and the angle AHG=CFE, *a* and the angle HAG=FCE. then AGHB shall be the right-lined figure sought. a 23. 1.

For the angle B *b* =D , and the angle BAH *b* = DCF. *c* wherefore the angle AHB = CFD. *b* also the angle HAG=FCE, and the angle AHG *b* = CFE. *c* wherefore the angle G=E , and the whole angle GAB *d* = ECD, and the whole angle GHB *d* = EFD. The polygones therefore are mutually equiangular. Moreover because the triangles are equiangular, therefore AB.BH *e* :: CD.DF; and AG. GH *e* :: CE. EF. Likewise AG.AH :: *e* CE.CF. and AH. AB :: CF. CD. *f* From whence by equality AG. AB :: CF. CD. After the same manner GH. HB :: EF. FD. Therefore the Polygones ABHG, CDFE are like and alike situate. *W.W. to be Done*. b *constr.* c 32. 1. d 2. ax. e 4. 6. f 22. 5. g 6. *def*. 5.

PROP. XIX.



Like triangles ABC, DEF *are in duplicate ratio of their homologous sides*, BC, EF.

a Let there be made BC. EF :: EF. BG, and let AG be drawn. Because that AB. a 11. 6.

DE

b cor.4 6. c constr. d 15.6. e 1.6 f 10.def 5. g 11. 5.

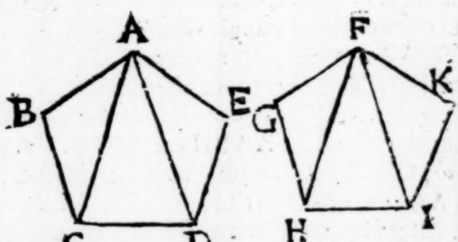
DE ^{b} :: BC. EF ^{c} :: EF. BG, and the angle B = E. ^{d} therefore is the triangle ABG = DEF. But the triangle ABC. ABG :: ^{e} BC. BG, and ^{f} $\frac{BC}{BG} = \frac{BC}{EF}$ twice; therefore $\frac{ABC}{ABG}$ that is $\frac{ABC}{DEF}$ $^g = \frac{BC}{EF}$ twice.

W.W. to be Dem.

Coroll.

Hence, If three right lines (BC, EF, BG) be proportionall, then as the first is to the third, so is a triangle made upon the first BC, to a triangle like and alike described upon the second EF : or so is a triangle described upon the second EF to a triangle like and alike described upon the third.

PROP. XX.



Like polygones ABCDE, FGHIK *are divided into equall triangles* ABC, FGH, *and* ACD, FHI, *and* ADE, FIK ; *both equall in number and homologous to the wholes* (ABC. FGH :: ABCDE. FGHIK :: ACD. FHI :: ADE. FIK.) *And the polygones* ABCDE, FGHIK *have a double ratio one to the other of what one homologous side* BC *hath to the other homologous side* GH.

a hyp. b 6.6.

1. For the angle B ^{a} = G, and AB. BC ^{a} :: FG. GH. ^{b} therefore the triangles ABC, FGH are

are equiangular. After the same manner are the triangles AED, FKI like. Whereas therefore the angle BCA *b* = GHF, and the angle ADE *b* = FIK, and the whole angles BCD, GHI, and the whole angles CDE, HIK are *c* equall, there remains the angle ACD *d* = FHI, and the angle ADC = FIH; *e* from whence also the angle CAD = HFI. therefore the triangles ACD, FHI are like. Therefore, &c. b 6. 6. c *hyp.* d 3 *ax.* e 32. 1.

2. Because that the triangles BCA, GHF are like, *f* therefore is $\frac{BCA}{GHF} = \frac{BC}{GH}$ twice. For the same reason f 19. 6.

is $\frac{CAD}{HFI} = \frac{CD}{HI}$ twice; lastly $\frac{DEA}{IKF} = \frac{DE}{IK}$ twice.

now whereas that BC. GH *g* :: CD. HI *g* :: DE. IK. *g hyp. & 16.*
k therefore is the triangle BCA. GHF :: CAD. HFI
:: DEA. IKF *h* :: the polygone ABCDE. FGHIK :: *h sch* 23. 5.
$\frac{BC}{GH}$ twice. k 12. 5.

Coroll.

1. Hence, If there be three right lines proportionall; then as the first is to the third, so is a polygone made upon the first to a polygone made on the second like and alike described; or so is a polygone described upon the second to a polygone made on the third like and alike described.

By which is found out a method of inlarging or diminishing any right-lined figure in a ratio given: As if you would make a pentagone quintuple of that pentagone whereof CD is the side, then betwixt AB and 5 AB find out a mean proportional,* upon this raise a pentagone like to that given, and it shall be quintuple of the pentagone given. * 13. 6.

2. Hence also, If the homologous sides of like figures be known, then will the proportion of the figures be evident, *viz.* by finding out a third proportional.

PROP. XXI.

Right-lined figures ABC, DIE *which are like to the same right-lined figure* HFG, *are also like one to the other.*

For the angle A a = H a = D; and the angle C
a 1.def.6. a = G a = E; and the angle B a = F a = I. Also
a AB.AC :: HF.HG :: DI.DE. & a AC.CB :: HG.
GF :: DE. EI. And AB. BC :: HF. FG :: DI. IE.
Therefore a ABC, DIE are like. *W. W. to be Dem.*

PROP. XXII.

If four right lines be proportionall (AB. CD :: EF. GH) *the right-lined figures also described upon them being like and in like sort situate, shall be proportionall* (ABI.CDK :: EM. GO.) *And if the right-lined figures described upon the lines, like and alike situate, be proportionall* (ABI. CDK :: EM. GO) *then the right lines also shall be proportionall* (AB. CD :: EF. GH.)

a 19. 6. 1. *Hyp.* $\frac{ABI}{CDK}$ a $=$ $\frac{AB}{CD}$ twice a $=$ $\frac{EF}{GH}$ twice a $=$ $\frac{EM}{GO}$.

b therefore ABI.CDK :: EM. GO.

2. *Hyp.* $\frac{AB}{CD}$ twice a $=$ $\frac{ABI}{CDK}$ b $=$ $\frac{EM}{GO}$ $=$ c $\frac{EF}{GH}$ twice.

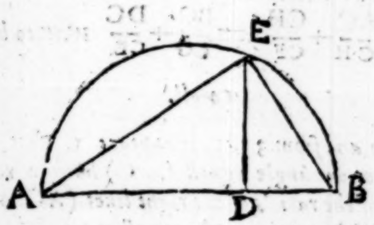
b hyp.
c 20. 6.
d cor.23.5. Therefore AB. CD :: EF. GH. *W. W. to be Dem.*

Schol.

Schol.

Hence is deduced the manner and reason of multiplying surd quantities. Ex.g. Let √5 be to be multiplyed into √3. I say that the product will be √15. For by the definition of multiplication it ought to be, as 1. √3 :: √5. to the product. Therefore by *this* q.1. q. √3 :: q. √5.q. of the product. That is 1.3. :: 5. to the square of the product. therefore the square of the product is 15. Wherefore √15 is the product of √3 into √5. *W.W.to be Dem.*

THEOREME.



If a right line AB *be cut any-wise in* D, *the rectangle comprehended under the parts* AD, DB *is a mean proportionall betwixt their squares. Likewise the rectangle comprehended under the whole* AB *and one part* AD, or DB, *is a mean proportionall betwixt the square of the whole* AB *and the square of the said part* AD, or DB. *Petr. Herig.*

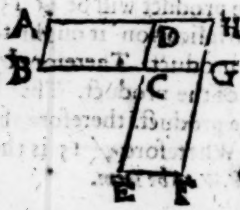
Upon the diameter AB describe a semicircle; from D erect a perpendicular DE meeting with the periphery in E. join AE, BE.

It's evident that AD.DE *a* :: DE.DB.*b* therefore ADq. DEq :: DEq. DBq. *c* that is, ADq. ADB :: ADB. DBq. *W.W.to be Dem.* a cor.8.6. b 22.6. c 17.6.

Moreover BA. AE :: *d* AE.AD. *e* therefore BAq. AEq :: AEq. ADq. *f* that is BAq. BAD :: BAD. ADq. After the same manner ABq. ABD :: ABD. BDq. *W.W.to be Dem.* d cor.8.6. e 22.6. f 17.6.

Or thus; suppose Z=A+E. It is manifest that Aq.AE :: *a* A.E :: *a* AE.Eq. also Zq.ZA :: *a* Z.A :: ZA.Aq.and Zq.ZE :: *a* ZE :: Z.E.Eq. a 1.6.

PROP. XXIII.



Equiangular parallelograms AC, CF, *have the ratio one to the other, which is compounded of their sides.* $\left(\frac{AC}{CF} = \frac{BC}{CG} + \frac{DC}{CE}\right)$

Let the sides about the a sch.15. equall angles C be *a* set in a direct line, and let the Pgr. CH be compleated. Then is the ratio of

b 20.def.5. c 1.6. $\frac{AC}{CF}{}^{b} = \frac{AC}{CH} + \frac{CH}{CF}{}^{c} = \frac{BC}{CG} + \frac{DC}{CE}$. *W.W.to be Dem.*

Coroll.

Andr.Tacq. 15.5. Hence, *and from* 34. 1. *it appears* 1. *That triangles which have one angle equall* (at C) *have a ratio compounded of the ratio's of the right lines,* (AC *to* CB, *and* LC *to* CF,) *containing the equall angle.*

* 35.1.

2. *That all rectangles, &* * *consequently all parallelograms, have their ratio one to the other compounded of the ratio's of base to base, and altitude to altitude.* After the like manner you may argue in triangles.

3. *From hence is apparent how to give the proportion of triangles and parallelograms.* Let there be two pgrs. X, and Z, whose bases are AC, CB, and altitudes CL, CF. Make CL. CF :: CB. O.

* 14.6 and 1.6 *then will it be X, Z :: AC. O.

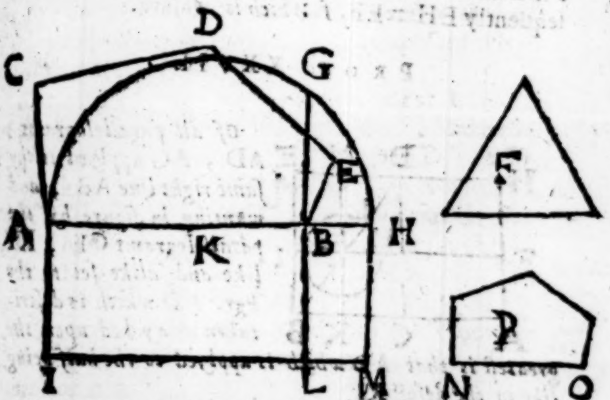
PROP. XXIV.

In every parallelogram ABCD, *the parallelograms* EG, HF *which are about the diameter* AC *are like to the whole, and also one to the other.*

For the Pgrs. EG, HF have each of them one angle common with the whole; a therefore they are equiangular to the whole, & also one to the other. Also both the triangles ABC, AEI, IHC a and the triangles ADC, AGI, IFC are equiangular mutually; b therefore AE.EI :: AB. BC, and b AE. AI :: AB. AC, and b AI. AG :: AC. AD. c Therefore by equality, AE. AG :: AB. AD. d Therefore the Pgrs. EG, BD are like. After the same manner are HE, BD like also. Therefore, &c.

a 29. 1.
b 4. 6.
c 22. 5.
d 1. def. 6.

PROP. XXV.



Unto a right-lined figure given ABEGD *to describe another figure* P *like and alike situate, which also shall be equall to another right-lined figure given* F.

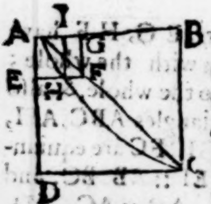
a Make the rectangle AL = ABEDC; b also upon BL make the rectangle BM = F; Betwixt AB and BH c find out a mean proportionall NO; Upon NO d make the polygone P like to the right-lined figure given ABEDC. I say the polygone P so made shall be equall to F that was given.

a 45. 1.
b 44. 1.
c 13. 6.
d 18. 6.

I 3 For

g cor. 20.6. For ABEDC (AL.) P :: e AB. BH :: f AL. BM,
f 1.6. Therefore P g = BM h = F. W. W. to be Done.
g 14 5.
h constr.

PROP. XXVI.



If from a Parallelogram AB CD, *be taken away another Parallelogram* AGFE, *like unto the whole, and in like sort set, having also an angle common with it* EAG; *then is that parallelogram about the same diagonall* AC *with the whole.*

If you denye AC to be the common diagonall, then let AHC be it, cutting EF in H, and let HI be drawn parallel to AE. Then are Pgrs EI, DB a like,
a 24. 6. b therefore AE.EH :: AD.DC :: c AB.EF. & d con-
b 1.def.6. sequently EH = EF. f *Which is Absurd.*
c hyp.
d 9 5.
f 9.ax.

PROP. XXVII.

Of all parallelograms AD, AG *applyed to the same right line* AB, *and wanting in figure by the parallelograms* CE, KI *like and alike set to the Pgr.* AD *which is described upon the half, the greatest is that* AD *which is applyed to the half being like to the defect* KI.

a 43. 1. For because that GE a = GC, and KI added in
b 2 ax. common. b thence is KE = CI c = AM. adde CG in
c 36. 1. common, d then is AG = to the Gnomon MBL. But
d 2.ax. the Gnomon MBL e ⊐ CE (AD.) Therefore AG
e 9.ax. ⊐ AD. *W.W. to be Done.*

PROP.

PROP. XXVIII.

Upon a right line given AB, *to apply a parallelogram* AP *equal to a right-lined figure given* C, *deficient by a parallelogram* ZR *which is like to another parallelogram given* D. * *Now it is requisite that the right-lined figure given* C, *whereunto the Pgr. to be applyed* AP *must be equall, be not greater then the Pgr.* AF *which is applyed upon the half line, the defects being like, namely the defect of the Pgr.* AF, *which is applyed to the half line, and the defect of the Pgr.* D *to be applyed whose defect is to be like to the Pgr. given.* a 27.6.

Bisect AB in E; upon EB a make the Pgr. EG like to the Pgr. D; and b let EG = C+I. c Make the Pgr. NT = I, and like to the Pgr. given D, or EG; Draw the diameter FB; Make FO = KN, and FQ = KT; thorough O and Q draw the parallels SR, QZ. Then is the Pgr. AP that which was sought. a 18.6. b sch.45.1. c 25.6.

For the Pgrs D, EG, OQ, NT, ZR are all d like one to the other, and the Pgr. EG = e NT + C = e OQ + C. f wherefore C = to the Gnomon OBQ g = AO + PG = h AO + EP = AP. d constr. & 24.6. e constr. f 3.ax. g 2.ax. h 43.1.

Which was to be Done.

PROP. XXIX.

Upon a right line given AB *to apply a pgr.* AN *equall to a right-lined figure given* C, *exceeding by a pgr.* OP, *which shall be like to another pgr given* D.

Bisect AB in E. Upon EB a make a Pgr. EG like to D, which was given. And b let the pgr. HK = EG + C, and like to D given, or EG. Make FEL = c IH; and c FGM = IK. Thorough L, M draw the parallels MN and RN and AR parallel to NM. produce ABP, GBO. Draw the diameter FBN. Then is AN the parallelogram required.

For the pgrs. D, HK, LM, EG are d like. e therefore the pgr. OP is like to the pgr. LM, or D. Also LM f = HK g = EG + C. g Therefore C = to the Gnomon ENG. But AL h = LB k = BM. l therefore C = AN. *W.W. to be Done.*

a 28.6.
b 25.6.
c 3.1.
d constr.
e 24.6.
f constr.
g 3 ax.
h 36.1.
k 43.1.
l 2. & 1. ax.

PROP. XXX.

To cut a finite right line given AB *according to extreme and mean ratio* (AB. AG :: AG. GB.)

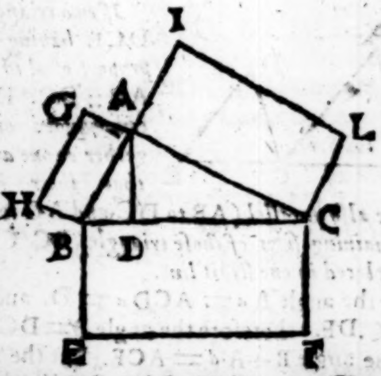
a Cut AB in G, in such wise that AB x BG = AGq.

b Then BA. AG :: AG. GB. *Which was to be Done.*

a 11.2.
b 17.6.

PROP.

PROP. XXXI.



In right-angled triangles BAC*; any figure* BF *described upon the side* BC *subtending the right-angle* BAC*, is equall to the figures* BG*,* AL *described upon the sides* BA*,* AC*, containing the right angle, like and alike situate to the former* BF.

From the right angle BAC let down a perpendicular AD. Because that DC.CA :: a CA.CB, b therefore AL. BF :: DC. CB. Also, because DB. BA :: a BA. BC, b therefore BG. BF :: DB.BC. c therefore AL+BG. BF :: DC+DB (BC.) BC. d Therefore AL+BG = BF. *W.W. to be Dem.* a cor. 8. 6. b cor. 20 6. c 24. 5. d schol. 14. 5.

Or thus: BG.BF :: e BAq. BCq. And e AL. BF :: ACq BCq. f therefore BG+AL.BF :: BAq+ACq. BCq. g Therefore whereas BAq+ACq = h BCq. thence is BG+AL = BF. *W.W. to be Dem.* e 22. 6. f 24. 5. g schol. 14. 5. h 47. 1.

Coroll.

From this proposition you may learn how to add or substract any like figures, by the same method that is used in adding and substracting of squares, in *Schol.* 47. 1.

PROP.

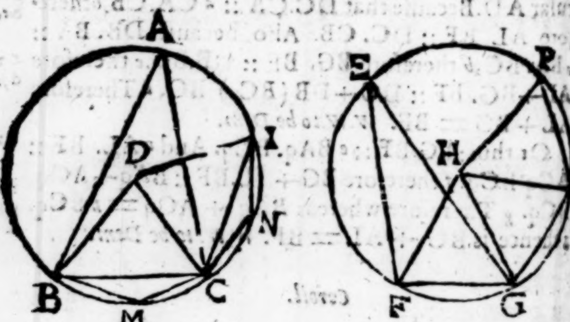
PROP. XXXII.

If two triangles ABC, DCE *having two sides proportional to two* (AB. AC :: DC. DE.) *be so compounded or set together at one angle* ACD *that their homologous sides be also parallel* (AB *to* DC, *and* AC *to* DE,) *then the remaining sides of those triangles* (BC, CE) *shall be found placed in one strait line.*

a 29. 1. b hyp. c 6. 6. d 2. ax. e 32. 1. f 1. ax. g 14. 1.

For the angle A a = ACD a = D. and AB. AC b :: DC. DE. c therefore the angle B = DCE. Therefore the angle B + A d = ACE. But the angle B + A + ACB e = 2 right. f therefore the angle ACE + ACB = 2 right. g therefore BCE is a right line. W.W. to be Dem.

PROP. XXXIII.



In equall circles DBCA, HFGP, *the angles* BDC, FHG *have the same ratio with their peripheries* BC, FG *on which they insist; whether the angles be set at the centers* (as BDC, FHG) *or at the circumferences,* A, E: *And in like sort are the Sectors* BDC, FHG, *because described upon the centers.*

Draw

Draw the right lines BC, FG. Make CI = CB, & GL = FG = LP and join DI, HL, HP.

The arch BC *a* = CI. *a* also the arch FG, GL, LP are equall; *b* therefore the angle BDC = CDI. *b* and the angle FGH = GHL = LHP. Therefore the arch BI is as multiplex of the arch BC, as the angle BDI is of the angle BDC. And in like manner is the arch FP as multiplex of the arch FG, as the angle FHP is of the ang. FHG. But if the arch BI $\sqsubset, =, \sqsupset$ FP, *c* then likewise is the angle BDI $\sqsubset, =, \sqsupset$ FHP. Therefore is the arch BC. FG :: *d* the angle BDC. FHG *e* :: $\underset{2}{\underline{BDC}}$. $\underset{2}{\underline{FHG}}$ *f* :: A. E. *W.W. to be Dem.* a 28. 3. b 27. 3. c 27. 3. d 6. def. 5. e 15. 5. f 10. 3.

Moreover, the angle BMC *g* = CNI; *h* and therefore the segment BCM = CIN. *k* Also the triangle BDC = CDI; *l* wherefore the Sector BDCM = CDIN. After the same manner are the sectors FHG, GHL, LHP equall one to the other. Therefore since accordingly as the arch BI $\sqsubset, =, \sqsupset$ FGP, so is likewise the sector BDI $\sqsubset, =, \sqsupset$ FHP; *m* thence shall be the sector BDC. FHG :: the arch BC. FG. *W.W. to be Dem.* g 27. 3. h 24. 3. k 4. 1. l 2. ax. m 6. def. 5.

Coroll.

1. Hence, *As sector is to sector, so is angle to angle.* 11. 5.

2. *The angle* BDC *in the center is to four right angles, as the arch* BC, *on which it insists, to the whole circumference.*

For as the angle BDC is to a right angle, so is the arch BC to a quadrant. Therefore BDC is so to four right angles as the arch BC is to four quadrants, that is, the whole circumference. Also, the angle A. 2 right :: the arch BC. periphery.

3. Hence, *The arches* IL, BC *of unequall circles which subtend equall angles, whether at the centers, as* IAL *and* BAC, *or at the periphery, are like.*

For

For IL. periph :: angle IAL (BAC.) 4 right. Also Arch BC. Periph :: angle BAC. 4 right. Therefore IL. periph :: BC periph. And consequently the arches IL and BC are like. Whence

4. *Two semidiameters* AB, AC *cut off like arches* IL, BC *from concentricall peripheries.*

The End of the sixth Book of Euclide's Elements.

THE SEVENTH BOOK OF EUCLIDE'S ELEMENTS.

Definitions.

I. UNity is that, by which every thing that is, is called One.

II. Number is a multitude composed of unities.

III. One number is a Part of another, the lesser of the greater, when the lesser measureth the greater.

Every part is denominated from that number, by which it measures the number whereof it is a part; as 4 is called the third part of 12, because it measures 12 by 3.

IV. But the lesser number is termed Parts, when it measureth not the greater.

All parts whatsoever are denominated from those two numbers, by which the greatest common measure of the two numbers measures each of them; as 10 is said to be $\frac{2}{3}$ of the number 15; because the greatest common measure, which is 5, measures 10 by 2, and 15 by 3.

V. A number is Multiplex (or Manifold) a greater in comparison of a lesser, when the lesser measureth the greater.

VI. An Even number is that which may be divided into two equall parts.

VII. But an Odde number is that which cannot be divided into two equall parts; or, that which differeth from an even number by an unitie.

VIII. A number Evenly Even is that which an even number measureth by an even number.

IX. But a number Evenly Odde is that which an even number measureth by an odde number.

X. A

X. A number Oddly Odd is that which an odd number measureth by an odd number.

XI. A Prime (or first) number is that, which is measured only by an unitie.

XII. Numbers Prime the one to the other, are such as onely an unitie doth measure, being their common measure.

XIII. A Composed number is that which some certain number measureth.

XIV. Numbers Composed the one to the other, are they, which some number, being a common measure to them both, doth measure.

In this, and the preceding definition, unitie is no number.

XV. One number is said to Multiply another, when the number multiplied is so often added to it self, as there are unities in the number multiplying, and another number is produced.

Hence in every multiplication a unitie is to the multiplier, as the multiplied is to the product.

Obs. *That many times, when any numbers are to be multiplied (as* A *into* B) *the conjunction of the letters denotes the Product: So* AB $=$ A$\times$B, *and* CDE $=$ C$\times$D $\times$ E.

XVI. When two numbers multiplying themselves produce another, the number produced is called a Plane number; and the numbers which multiplied one another, are called the Sides of that number: *So* 2 (C) $\times$ 3 (D) $=$ 6 $=$ CD *is a plane number.*

XVII. But when three numbers multiplying one another produce any number, the number produced is termed a Solid number; and the numbers multiplying one another, are the sides thereof: *So* 2 (C) $\times$ 3 (D) $\times$ 5 (E) $=$ 30 $=$ CDE *is a solid number.*

XVIII. A Square number is that which is equally equall; or, which is conteined under two equall numbers. *Let* A *be the side of a square; the square is thus noted,* AA, *or* Aq.

XIX. A Cube is that number which is equally equall

quall equally; or, which is conteined under three equall numbers. *Let* A *be the side of a Cube; the Cube is thus noted*, AAA, *or* Ac.

In this definition, and the three foregoing, unitie is a number.

XX. Numbers are proportionall, when the first is as multiplex of the second as the third is of the fourth; or, the same part; or, when a part of the first number measures the second, and the same part of the third measures the fourth, equally: and on the contrary. *So* A.B :: C. D, *that is*, 3.9 :: 5.15.

XXI. Like Planes, and solid numbers are they, which have their sides proportionall: *Namely, not all the sides, but some.*

XXII. A Perfect number is that which is equall to its own parts.

As 6. & 28. *But a number that is lesse then it's parts is called an* Abounding *number; and a greater a* Diminutive: *so* 12 *is an abounding*, 15 *a diminutive number.*

XXIII. One number is said to measure another, by that number, which, when it multiplies, or is multiplied by it, it produceth.

In Division, a unitie is to the quotient as the divisor is to the dividend. Note, *that a number placed under another with a line between them, signifies division*: So $\frac{A}{B}$ = A divided by B, & $\frac{CA}{B}$ = C x A divided by B.

Two numbers are called Termes or Roots of Proportion, lesser then which cannot be found in the same proportion.

Postulates, or Petitions.

1. THat numbers equall or manifold to any number may be taken at pleasure.

2. That a greater number may be taken then any number whatsoever.

3. That Addition, Substraction, Multiplication,

Division, and the Extractions of roots or sides of square and cube numbers, be also granted as possible.

Axiomes.

1. WHatsoever agrees with one of many equall numbers, agrees likewise with the rest.

2. Those parts that are the same to the same part, or parts, are the same amongst themselves.

3. Numbers that are the same parts of equall numbers, or of the same number, are equall amongst themselves.

4. Those numbers, of whom the same number, or equall numbers, are the same parts, are equall amongst themselves.

5. Unitie measures every number by the unities that are in it; that is, by the same number.

6. Every number measures it self by a unitie.

7. If one number, multiplying another, produce a third, the multiplier shall measure the product by the multiplied; and the multiplied shall measure the same by the multiplier.

Hence, No prime number is either a plane, solid, square, or cube number.

8. If one number measure another, that number by which it measureth shall measure the same by the unities that are in the number measuring, that is, by the number it self that measures.

9. If a number measuring another, multiply that by which it measureth, or be multiplied by it, it produceth the number which it measureth.

10. How many numbers so ever any number measureth, it likewise measures the number composed of them.

11. If a number measure any number, it also measureth every number which the said number measureth.

12. A number that measures the whole & a part taken away, doth also measure the residue.

PROPOSITION I.

A ..... E ... G . B   8 5 3
C ... F .. D $\frac{5}{3}$ $\frac{3}{2}$ $\frac{2}{1}$
H - - -

If two unequall numbers AB, CD, *being given, the lesser* CD *be continually taken from the greater* AB *(and the residue* EB *from* CD, &c.) *by an alternate substraction, and the number remaining do never measure the precedent, till the unitie* GB *be taken; then are the numbers which were given* AB, CD, *prime the one to the other.*

If you deny it, let AB, CD have a common measure, namely the number H. Therefore H measuring CD, doth a also measure AE; and consequently the a 11. ax 7.
remainder EB; a therefore it likewise measures CF, and b so the remainder FD; a wherefore it also measures EG. But it measured the whole EB, and b therefore it must measure that which remaineth GB, a unitie, it self being a number. c *Which is Absurd.* b 12. ax 7. c 9. ax. 1.

PROP. II.

9 6
A .... E ...... B   15 9 6
6 3
C ...... F ... D $\frac{9}{6}$ $\frac{6}{3}$ $\frac{3}{0}$
G - - -

Two numbers AB, CD *being given, not prime the one to the other, to find out their greatest common measure* FD.

Take the lesser number CD from the greater AB as often as you can. If nothing remains, a it is manifest that CD is the greatest common measure. But if there remains something (as EB) then take it out of CD, and the residue FD out of EB, and so forward till some number (FD) measure the said EB (b for this will be, before you come to a unitie.) FD shall be the greatest common measure. a 6. ax. 7. 1 b 1. 7.

For FD c measures EB, and d therefore also CF; and e consequently the whole CD; d therefore likewise AE; and so measures the whole AB. Wherefore c constr. d 11 ax. 7. e 12. ax. 7.

 it

it is evident that FD is a common measure. If you deny it to be the greatest, let there be a greater (G;) then whereas G measureth CD, it d must likewise measure AE, e & the residue EB, d as also CF, e and by consequence the residue FD, g the greater the lesse. h *Which is absurd.*

d 11.ax.7.
e 12.ax.7.
g suppos.
h 9.ax.1.

Coroll.

Hence, A number that measures two numbers, does also measure their greatest common measure.

Prop. III.

A 12	
B 8	
	D 4
C 6	
	E .. 2
F - - -	

Three numbers being given, A, B, C, *not prime to one another*, *to find out their greatest common measure* E.

Find out D the greatest common measure of the two numbers A, B. If D measures C the third, it is clear that D is the greatest common measure of all the three numbers. If D does not measure C, at least D and C will be composed the one to the other, by the Coroll. of the Prop. preceding. Therefore let E be the greatest common measure of the said numbers D and C, and it appears to be the number required.

a constr.
b 11.ax.7.
c cor. 2.7.

For E a measures C and D, and D measures A and B; therefore b E measures each of the numbers A, B, C: neither shall any greater (F) measure them; for if you affirm that, c then F measuring A and B, does likewise measure D their greatest common measure; and in like manner, F measuring D and C, does also measure E c their greatest common measure, d the greater the lesse. e *Which is absurd.*

d suppos.
e 9.ax.1.

Coroll.

Hence, A number that measures three numbers, does also measure their greatest common measure.

PROP.

PROP. IV.

A...... 6
B....... 7
B.................. 18
B......... 9

Every lesse number A *is of every greater* B *either a part or parts.*

If A and B be prime to one another, *a* A shall be as many parts of the number B, as there are unities in A (as $6 = \frac{6}{7}7$.) But if A measures B, it is *b* plain that A is a part of B (as $6 = \frac{1}{3}18$.) Lastly, if A and B be otherwise composed to one another, *c* the greatest common measure shall determine how many parts A does contain of B; as $6 = \frac{2}{3}9$.

a 4. *def.* 7.
b 3. *def.* 7.
c 4. *def.* 7.

PROP. V.

A...... 6 D.... 4
6 6 4 4
B...... G...... C 12. E.... H.... F 8.

If a number A *be a part of a number* BC, *and another number* D *the same part of another number* EF; *then both the numbers together* (A + D) *shall be the same part of both the numbers together* (BC + EF,) *which one number* A *is of one number* BC.

For if BC be resolved into it's parts BG, GC, equall to A; and EF also into its parts EH, HF equall to D; *a* the number of parts in BC shall be equall to the number of parts in EF. Therefore since A + D *b* = BG + EH = GC + HF, thence A + D shall be as often in BC + EF, as A is in BC. *Which was to be Demonstrated.*

a *hyp.*
b *const.* and 2. *ax.* 1.

Or thus. Let $a = \frac{x}{2}$, and $b = \frac{y}{2}$. then $2a = x$, and $2b = y$. wherefore $2a + 2b = x + y$. therefore $a + b = \frac{x+y}{2}$. *W. W. to be Dem.*

c 2. *ax.* 1.

PROP. VI.

3 3 4 4
A ... G ... B 6 D .... H .... E 8
C .... 9 F .... 12.

If a number AB *be parts of a number* C, *and another number* DE *the same parts of another number* F; *then both numbers together* AB+DE *shall be of both numbers together* C+F *the same parts, that one number* AB *is of one number* C.

Divide AB into its parts AG, GB; and DE into its parts DH, HE. The multitude of parts in both AB, DE is equall by supposition; Wherefore since AG *a* is the same part of the number C, that DH is of the number F; therefore AG+DH *b* shall be the same part of the compounded number C+F, that one number AG is of one number C. *b* In like manner GB+HE is the same part of the said C+F, that one number GB is of one number C. *c* Therefore AB+DE is the same parts of C+F, that AB is of C. *W. W. to be Dem.*

a hyp.
b 5. 7.
c 2. ax. 7.

Or thus. Let $a = \frac{2}{3} x$, and $b = \frac{2}{3} y$, and $x + y = g$. then, because $3a = 2x$, and $3b = 2y$, is $3a + 3b = 2x + 2y = 2g$. therefore $a + b = \frac{2}{3} g = \frac{2}{3} : x + y$.

PROP. VII.

5 3
A ..... E ... B 8
6 10 6
G ...... C .... F ...... D 16

If a number AB *be the same part of a number* CD, *that a part taken away* AE *is of a part taken away* CF; *then shall the residue* EB *be the same part of the residue* FD *that the whole* AB *is of the whole* CD.

a Let EB be the same part of the number GC that AB is of CD, or AE of CF. *b* therefore AE+EB is the same part of CF+GC that AE is of CF, or AB of CD. *c* therefore GF=CD. Take away CF common to both, & *d* there remains GC=FD. *e* Wherefore EB is the same part of the residue FD (GC) that the whole AB is of the whole CB. *Which was to be Dem.*

a 1. post. 7.
b 5. 7.
c 6. ax. 1.
d 3. ax. 1.
e 2. ax. 7.

Or thus. Let $a+b=x$; and $c+d=y$; and $x=3y$, in like manner as $a=3c$; I say $b=3d$. For $3c+3d$ *f* $=3y=x$ *g* $=a+b$. take away from both $3c$ *g* $=a$, and *h* there remains $3d=b$. *W.W. to be Dem.* f 1.2. g hyp.

PROP. VIII.

```
  6      2    4      2   2
A ...... H .. G .... E .. L .. B 16
       18               6
C ................ F ...... D 24
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If a number AB *be the same parts of a number* CD, *that a part taken away* AE *is of a part taken away* CF; *the residue also* EB *shall be of the residue* FD *the same parts, that the whole* AB *is of the whole* CD.

Divide AB into AG, GB, parts of the number CD; also AE into AH, HE, parts of the number CF; and take $GL=AH=HE$. *a* wherefore $HG=EL$. And because *b* $AG=GB$, *c* therefore $HG=LB$. Now whereas the whole AG is the same part of the whole CD that the part taken away AH is of the part taken away CF, *d* the residue HG or EL shall be the same part also of the residue FD that AG is of CD. In like manner, because GB is the same part of the whole CD, that HE or GL are of CF, *d* therefore the residue LB shall be the same part of the residue FD that GB is of the whole CD. Therefore $EL+LB$ (EB) is the same parts of the residue FD, that the whole AB is of the whole CD. *W.W. to be Dem.*

a 3. ax. 1. b constr. c 3 ax 1. d 7. 7.

Or thus more easily. Let $a+b=x$, and $c+d=y$. Also $y=\frac{2}{3}x$ as well as $c=\frac{2}{3}a$; or, *e* which is the same, $3y=2x$; and $3c=2a$. I say $d=\frac{2}{3}b$. For $3c+3d$ *f* $=3y=2x$ *f* $=2a+2b$. *g* Therefore $3c+3d=2a+2b$. take away from each $3c$ *h* $=2a$, and *k* there remains $3d=2b$. *l* therefore $d=\frac{2}{3}b$. *Which was to be Dem.*

e 9 ax. 7 f 1. 2. g 1. ax. 1. h hyp k 3. ax. 1. l 8 ax. 7.

PROP. IX.

A 4
4 4
B .... G .... C 8
5 D 5
5 5
E ..... H ..... F 10

If a number A *be a part of a number* B C , *and another number* D *the same part of another number* EF ; *then alternately what part or parts the first* A *is of the third* D, *the same part or parts shall the second* BC *be of the fourth* EF.

A is supposed $\sqsupset$ D. therefore let BG , GC, and EH, HF, parts of the numbers BC , EF be equall; BG and GC to A; and EH, HF to D. The multitude of parts is put equall in both. But it is clear that BG is a the same part or parts of EH, that G C is of H F; b wherefore BC (BG+GC) is the same part or parts of EF (EH + HF) that BG alone (A) is of EH alone (D.) *Which was to be Dem.*

a 1.ax.7. and 4 7.
b 5, & 6 7.

Or thus. Let $a = \frac{b}{3}$, and $c = \frac{d}{3}$; or $3a = b$, and $3c = d$. then is $\frac{c}{a} * = \frac{3c}{3a} = \frac{d}{b}$.

* 15, 5.

PROP. X.

A .. G .. B 4
C 6
5 5
D ..... H ..... E 10
F 15

If a number AB *be parts of a number* C , *and another number* DE *the same parts of another number* F; *then alternately, what parts or part the first* AB *is of the third* DE, *the same parts or part shall the second* C *be of the fourth* F.

AB is taken $\sqsupset$ DE, and C $\sqsupset$ F. Let AG, GB, and DH, HE be parts of the numbers C and F, *viz.* as many in AB, as in DE. It is manifest that AG is the same part of C, that DH is of F. a whence alternately AG is of DH, and likewise GB of HE, & b so conjointly AB of DE the same part, or parts, that C is of F. *Which was to be Dem.*

a 9. 7.
b 5. and 9 7.

Or thus. Let $a = \frac{2b}{3}$, & $c = \frac{2d}{3}$; or $3a = 2b$, & $3c = 2d$. Then is $\frac{c}{a} = \frac{3c}{3a} = \frac{2d}{2b} = \frac{d}{b}$.

PROP.

PROP. XI.

4 3
A .... E ... B 7
8 6
C .... F ...... D 14

If a part taken away AE *be to a part taken away* CF, *as the whole* AB *is to the whole* CD, *the residue also* EB *shall be to the residue* FD, *as the whole* AB *is to the whole* CD.

First, let AB be ⊐ CD; *a* then AB is either a part or parts of the number CD; and likewise AE is *b* the same part or parts of CF; *c* therefore the residue EB is the same part or parts of the residue FD that the whole AB is of the whole CD. *b* & so AB. CD :: EB. FD. But if AB be ⊏ CD, then according to what is already shewn, will CD. AB :: FD. EB. therefore by inversion AB. CD :: EB. FD.

a 4. 7.
b 10. def 7.
c 7. or 8. 7.

PROP. XII.

A, 4. C, 2. E, 3.
B, 8. D, 4. F, 6.

If there be divers numbers, how many soever, proportionall (A.B :: C. D :: E. F;) *then as one of the antecedents* A *is to one of the consequents* B, *so shall all the antecedents* (A + C + E) *be to all the consequents* (B+D+F.)

First, let A, C, E, be ⊐ B, D, F; then (because of the same proportions) *a* shall A be the same part or parts of B that C is of D; *b* and likewise conjointly A+C shall be the same part or parts of B+D that A alone is of B alone. In the like manner A+C+E is the same part or parts of B + D + F that A is of B. *c* Therefore A+C+E. B+D+F :: A. B. But if A, C, E, be put greater then B, D, F, the same may be shewn by inversion.

a 10 def 7.
b 5. & 6. 7.
c 10. def 7.

Prop. XIII.

A, 3. C, 4. *If there be four numbers proportionall*
B, 5. D, 12. (A. B :: C. D.) *then alternately they shall also be proportionall,* (A.C :: B.D.)

First let A and C be ⊐ B and D, and A ⊐ C. By a 10. def. 7. reason of the same proportion a shall A be the same b 9. and 10. 7. part or parts of B, that C is of D. b Therefore alternately A is the same part or parts of C that B is of D. and so A. C :: B. D But if A be ⊏ C, and A and C supposed ⊐ B and D, the matter will be the same by inverting the proportions.

Prop. XIV.

A, 9. D, 6. *If there be numbers, how many soever,*
B, 6. E, 4. A, B, C, *and as many more equall to*
C, 3. F, 2. *them in multitude, which may be compared two and two in the same proportion* (A.B :: D.E. *and* B.C :: E.F;) *they shall also, of equality, be in the same proportion* (A.C :: D.F.)

For because A.B :: D.E. a therefore alternately is a 13. 7. A.D :: B. E :: a C. F. a and so again by changing A.C :: D.F. *W.W. to be Dem.*

Prop. XV.

1. D..
B ... 3. E ...... 6. *If a unite measure any number* B, *and another number* D *do equally measure some other number* E; *alternately also shall a unite measure the third number* D, *as often as the second* B *doth the fourth* E.

For seeing 1 is the same part of B, that D is of E; a 9. 7. a therefore alternately shall 1 be the same part of D, that B is of E. *W.W. to be Dem.*

PROP. XVI.

B, 4.	A, 3.
A, 3.	B, 4.
AB, 12.	BA, 12.

If two numbers A, B, *multiplying themselves the one into the other, produce any numbers* AB, BA; *the numbers produced* AB *and* BA *shall be equall the one to the other.*

For because AB = A x B, a therefore shall 1 be as often in A, as B in AB, b and by consequence alternately 1 shall be as often in B as A in AB. But for that BA = B x A, a therefore shall 1 be as often in B, as A in BA. therefore as often as 1 is in AB, so often is 1 in BA. and c so AB = BA. *W.W. to be Dem.*

a 15. def. 7.
b 15. 7.
c 4. ax. 7.

PROP. XVII.

A, 3.	
B, 2.	C. 4.
AB, 6.	AC, 12.

If a number A *multiplying two numbers* B, C, *produce other numbers* AB, AC; *the numbers produc'd of them shall be in the same proportion that the numbers multiplyed are.* (AB. AC :: B. C.)

For being AB = A x B, a therefore shall 1 be as often in A, as B in AB. a Likewise because AC = A x C, therefore shall 1 be as often in A, as C in AC. and so also B as often in AB as C in AC. b wherefore B. AB :: C. AC. c and therefore also alternately B. C :: AB. AC. *W.W. to be Dem.*

a 15. def. 7.
b 20. def. 7.
c 13. 7.

PROP. XVIII.

C, 5.	C, 5.
A, 3.	B, 9.
AC, 15.	BC, 45.

If two numbers A, B *multiplying any number* C, *produce other numbers* AC, BC; *the numbers produced of them shal be in the same proportion that the numbers multiplying are* (A. B :: AC. BC.)

For a AC = CA, and BC a = CB; so the same C multiplying A and B produceth AC and BC. b therefore A. B :: AC. BC. *W.W. to be Dem.*

a 16. 7.
b 17. 7.

Schol.

Schol.

Hence is deduced the vulgar manner of reducing fractions ($\frac{3}{5}, \frac{7}{9}$) to the same denomination. For multiply 9 both by 3 and 5, and they produce $\frac{27}{45} = \frac{3}{5}$ because by this, 3. 5 :: 27. 45. Likewise multiply 5 by 7 and 9; there arises $\frac{35}{45} = \frac{7}{9}$; because 7.9 :: 35. 45.

Prop. XIX.

A, 4. B, 6. C, 8. D, 12.
AD, 48. BC, 48.

If there be four numbers in proportion (A.B :: C.D) *the number produced of the first and fourth* (AD) *is equall to the number which is produced of the second and third* (BC.) *And if the number which is produced of the first and fourth* (AD) *be equall to that produced of the second & third* (BC) *those four numbers shall be in proportion* (A. B :: C. D.)

a 17. 7. b hyp. c 18. 7. d 9. 5. e hyp. f 7. 5. g 17. 7. h 18. 7. k 11. 5.

1. *Hyp.* For AC.AD a :: C.D b :: A.B c :: AC.BC. d therefore AD = BC. *W.W. to be Dem.*

2. *Hyp.* Because e AD = BC, therefore AC. AD f :: AC. BC. But AC. AD g :: C. D. and AC. BC h :: A.B. k therefore C.D :: A.B. *W.W. to be Dem.*

Prop. XX.

A.	B.	C.
4.	6.	9.
AC, 36.	BB, 36.	
	D, 6.	

If there be three numbers in proportion (A. B :: B. C.) *the number contained under the extremes* (AC) *is equall to the square made of the middle* (BB.) *And if the number contained under the extremes be equall to that* (Bq,) *produced of the middle, those three numbers shall be in proportion* $\left(\frac{A}{B} :: \frac{B}{C.}\right)$

a 1. ax. 7. b 19. 7.

1. *Hyp.* For take D = B. a therefore A B :: D (B.) C. b wherefore AC = BD, or BB. *Which was to be Dem.*

2. *Hyp.*

2.*Hyp*. Because AC [c] = BD, [d] therefore A. B :: D (B) C. *W.W. to be Dem.* c hyp. d 19.7.

PROP. XXI.

A... G.. B 5. E .... 10.
C.. H. D 3. F ......, 6.

Numbers AB, CD, *being the least of all that have the same proportion with them* (E, F) *doe equally measure the numbers* E, F, *having the same proportion with them; the greater* AB *the greater* E, *and the lesser* CD *the lesser* F.

For AB.CD [a] :: E.F. [b] therefore alternately AB. E :: CD.F. [c] therefore AB is the same part or parts of E that CD is of F. Not parts; for if so, then let AG, GB be parts of the number E; and CH, HD, parts of the number F. [c] therefore AG. E :: CH. F. and by inversion AG. CH [d] :: E. F [e] :: AB. CD. therefore AB, CD are not the least in their proportion; which is contrary to the hypothesis. Therefore, &c.

a hyp. b 13.7. c 20.def.7. d 13.7. e hyp.

PROP. XXII.

A, 4. D, 12.
B, 3. E, 8.
C, 2. F, 6.

If there be three numbers A, B, C; *and other numbers equall to them in multitude*, D, E, F; *which may be compared two & two in the same proportion : and if also the proportion of them be perturbed* (A.B :: E. F. *and* B.C :: D.E.) *then of equality they shall be in the same proportion* (A.C :: D. F.)

For because A.B [a] :: E.F, therefore shall AF = BE; and because B. C :: [a] D. E, [b] therefore BE = CD. [c] and consequently AF = CD. [d] Wherefore A. C :: D.F. *W. W. to be Dem.*

a hyp. b 19.7. c 1.ax.1. d 19.7.

PROP.

Prop. XXIII.

A, 9. B, 4.
C - - - - D - - -
E - -

Numbers prime the one to the other, A, B, are the least of all numbers that have the same proportion with them.

If it be possible, let C and D be lesse then A and B, and in the same proportion; a therefore C measures A equally as D measures B, namely by the same number F; and so C shall be b as often in A as 1 is in E; c likewise alternately, E as often in A as 1 in C. By the like inference as many times as 1 is in D, so many times shall E be in B. Therefore E measures both A and B; which consequently are not prime the one to the other, *contrary to the Hypoth.*

a 21. 7.
b 23. def. 7.
c 15. 7.

Prop. XXIV.

A, 9. B, 4.
C - - -
D - - - E - -

Numbers A, B, being the least of all that have the same proportion with them, are prime the one to the others.

If it be possible, let A and B have a common measure C; and let the same measure A by D, and B by E; a therefore CD = A, b and CE = B. b Wherefore A.B :: D.E. But D and E are lesser then A and B, as being but parts of them. Therefore A and B are not the least in their proportion, *against the Hypoth.*

a 9. ax. 7.
b 17. 7.

Prop. XXV.

A, 9. B, 4.
C, 3. D - -

If two numbers A, B, be prime the one to the other, the number C measuring one of them A, shall be prime to the other number B.

For if you affirme any other D to measure the numbers B and C, a then D measuring C does also measure A; and consequently A and B are not prime the one to the other: *Which is against the Hypothesis.*

a 11. ax. 7.

Prop.

PROP. XXVI.

A, 5. C, 8.
B, 3.
$\overline{AB}$, 15. E - - - -
F - - - -

If two numbers, A, B, be prime to any number C, the number also produced of them AB shall be prime to the same C.

If it be possible, let the number E be a common measure to AB, and C; and let $\frac{AB}{E}$ be = F; a thence AB = EF; b wherefore also E.A :: B.F. But because A is prime to C, which E measures, c therefore E and A are prime to one another, d and so least in their own proportion, e and consequently they must measure B and F; namely F shall measure B, and A shall measure F. Therefore seeing E measures both B and C, they shall not be prime to one another: *contrary to the Hypoth.*

a 9. ax. 7.
b 19. 7.
c 25. 7.
d 23. 7.
e 21. 7.

PROP. XXVII.

A, 4. B, 5.
Aq, 16.
D, 4.

If two numbers A, B, be prime to one another, that also which is produc'd of one of them (Aq) shall be prime to the other B.

Take D = A; a therefore each of D, and A are prime to B. b wherefore A D or Aq is prime to B. *W.W. to be Dem.*

a 1 ax 7.
b 26. 7.

PROP. XXVIII.

A, 5. C, 4.
B, 3. D, 2.
$\overline{AB}$, 15. $\overline{CD}$, 8.

If two numbers A, B be prime to two numbers C, D each to either of both, the numbers also produced of them AB, CD shall be prime to one another.

For being A and B are prime to C, a therefore shall AB also be prime to the same. And by the same reason

a 26. 7.

b 26. 7. son shall AB be prime to D. *b* Therefore AB is prime to CD. *W.W. to be Dem.*

Prop. XXIX.

A, 3.	B, 2.
Aq, 9.	Bq, 4.
Ac, 27.	Bc, 8.

If two numbers A, B, *be prime to one another, and each multiplying himself produce another number* (Aq, *and* Bq;) *then the numbers produced of them* (Aq, Bq) *shall be prime to one another. And if the numbers given at first,* A, B, *multiplying the said produced numbers* (Aq, Bq) *produce others* (Ac, Bc) *those numbers also shall be prime to one another: and this shall ever happen about the extremes.*

For because A is prime to B, *a* therefore Aq shall
a 27. 7. be prime to B. and Aq being prime to B, *a* therefore Aq shall also be prime to Bq. Again, because A is as well prime to B and Bq, as Aq is to the said B and
b 28. 7. Bq, *b* therefore shall A x Aq, that is, Ac, be prime to B x Bq, that is, to Bc: And so forth of the rest.

Prop. XXX.

S 5
A .... B ..... C 13. D - - - -

If two numbers AB, BC, *be prime the one to the other; then both added together* (AC) *shall be prime to either of them* AB, BC. *And if both added together* AC *be prime to any one of them* AB, *the numbers also given in the beginning* AB, BC, *shall be prime to one another.*

1. *Hyp.* For if you would have AC, AB to be com-
a 12. ax. 7. posed, let D be the common measure: *a* this shall measure the residue BC: and therefore AB, BC, are not prime to one another; *which is against the hypoth.*

2. *Hyp.* AC, AB being taken for prime to one another, let D be the common measure of AB, BC.
b 10. ax. 7. *b* But seeing that measures the whole AC, therefore AC, AB, are not prime to one another; *contrary to the hypoth.*

Co-

Coroll.

Hence, A number, which being compounded of two, is prime to one of them, is also prime to the other.

PROP. XXXI.

A,5. B,8. *Every prime number* A *is prime to every number* B, *which it measureth not.*

For if any common measure doth measure both, A,B,[a] then A will not be a prime number; *contrary to the Hyp.* a 11. def. 7.

PROP. XXXII.

A,4. D,3. *If two numbers* A, B, *multiplying*
B,6. E,8. *one another produce another* AB, *and*
AB,24. *some prime number* D *measure the number produced of them* AB; *then shall it also measure one of those numbers,* A *or* B, *which were given at the beginning.*

Suppose the number D not to measure the number A, and let $\frac{AB}{D}$ be = E.[a] then AB=DE;[b] whence D.A :: B.E.[c] But D is prime to A;[d] therefore D and A are the least in their proportion;[e] and consequently D measures B as often as A measures E. *Which was to be Dem.* a 9. ex. 7. b 19. 7. c hyp. and 31. 7. d 23. 7. e 21. 7.

PROP. XXXIII.

A,12. *Every composed number* A *is measured*
B,2. *by some prime number* B.

Let one or more numbers[a] measure A, of which let the least be B; that shall be a prime number: for if it be said to be composed, then some[a] lesser number shall measure it, [b] which shall also consequently measure A. Wherefore B is not the least of them which measure A. *contrary to the Hyp.* a 13. def. 7. b 11. ax. 7.

PROP.

PROP. XXXIV.

A, 9. *Every number A, is either a prime, or measured by some prime number.*

For A is necessarily either a prime or a composed number. If it be a prime, 'tis that we affirm. If composed, [a] then some prime number measureth it. *W.W. to be Dem.*

a 33. 7.

PROP. XXXV.

A, 6. B, 4. C, 8.
D, 2.
E, 3. F, 2. G, 4.

H -- I -- K ----
L ---

How many numbers soever A, B, C, *being given, to find the least numbers* E, F, G, *that have the same proportion with them.*

a 23. 7.
b 3. 7.

If A, B, C, be prime to one another, [a] they shall be the least in their proportion. If they be composed, [b] let their greatest common measure be D, which let measure them by E, F, G. These are then least in the proportion A, B, C.

c 9. ax. 7.
d 17. 7.

For D x E, F, G, [c] produceth ABC, [d] therefore they are all in the same proportion. But allow other numbers H, I, K to be the least in the same proportion; [e] which shall therefore equally measure A, B, C, namely by the number L. [f] therefore L x H, I, K, shall produce A, B, C, [g] and consequently ED = A = HL. [h] from whence E. H :: L. D. But E [k] ⊏ K; [l] therefore L ⊏ D, and so D is not the greatest common measure of A, B, C. *Which is against the Hypoth.*

e 21. 7.
f 9. ax. 7.
g 1. ax. 1.
h 19. 7.
k suppos.
l 20. def. 7.

Coroll.

Hence, The greatest common measure of how many numbers soever do's measure them by the numbers which are least of all that have the same proportion with them. Whereby appears the vulgar method of reducing fractions to the least termes.

PROP.

PROP. XXXVI.

Two numbers being given A, B, to find out the least number which they measure.

A, 5. B, 4.
AB, 20.
D - - - - -
E - - - F - - -

1. *Case.* If A and B be prime the one to the other, AB is the number required. For it is manifest that A and B measure AB. If it be possible, let A and B measure some other number D ⊐ AB, if you please by E, and F. [a] therefore AE = D = BF, [b] and so A.B :: F.E. But because A and B [c] are prime the one to the other, [d] & so least in their proportion, A shall [e] equally measure F as B does E. But B.E [f] :: B. AE (D) [g] Therefore AB shall also measure D, which is lesse then it self. *Which is Absurd.* a 9 ax. 7 and 1. ax. 1. b 19. 7. c hyp. d 23. 7. e 21. 7. f 17. 7. g 10. def. 7.

A, 6. B, 4. F - - - -
C, 3. D, 2. G - - - H - - -
AD, 12.

2. *Case.* But if A and B be composed one to another, [h] let there be found C and D the least in the same proportion. [k] therefore AD = BC; and AD or BC shall be the number sought for. h 35 7. k 19. 7.

For it is [l] plain that B and D doe measure AD or BC. Conceive A and B to measure F ⊐ AD, namely A by G, and B by H. [m] therefore AG = F = BH. [n] whence A. B :: H. G [o] :: C. D. [p] and consequently C equally measures H as D does G. But D. G [q] :: AD. AG (F.) therefore AD [r] measures F, the greater the lesse. *Which is Absurd.* l 7. ax 7. m 9 ax 7. n 19. 7. o constr. p 21. 7. q 17. 7 r 10. def 7.

Coroll.

Hence, If two numbers multiply the least that are in the same proportion, the greater the lesse, and the lesse the greater, the least number which they measure shall be produced.

Prop. XXXVII.

A, 2. B, 3.
E 6
C - - - - F - - - D

If two numbers A, B, *measure any number* CD, *the least number which they measure* E *shall also measure the same* CD.

If you deny it, take E from CD as often as you can, and leave FD $\sqsupset$ E. therefore seeing A and B a measure E, b and E measures CF, c likewise A and B will measure CF. But a they measure the whole CD; d therefore also they measure the residue FD; and consequently E is not the least which A and B measure: *Contrary to the Hypoth.*

a hyp.
b constr.
c 11. ax. 7.
d 12. ax 7.

Prop. XXXVIII.

A, 3. B, 4. C, 6.
D, 12.

Three numbers being given A, B, C, *to find out the least which they measure.*

a Find D to be the least that two of them A and B do measure; which if the third C do also measure, it is manifest that D is the number sought for. But if C do not measure D, let E be the least that C and D do measure. E shall be the number required.

a 36. 7.

A, 2. B, 3. C, 4.
D, 6. E, 12.
F - - -

For it appears by the 11. ax. 7. that A, B, C measure E; and it is easily shewn that they measure no other lesse F. For if you affirm they do, b then D measures F, b and consequently E measures the same F, the greater the lesse. *Which is Absurd.*

b 37. 7.

Coroll.

Hence it appears that, If three numbers measure any number, the least also, which they measure, shall measure the same.

PROP. XXXIX.

A, 12.
B, 4. C, 3.

If any number B *measure a number* A, *the number measured* A, *shall have a part* C *denominated of the number measuring* B.

For because $\frac{A}{B}$ [a] = C, [b] shall A = BC. [c] therefore $\frac{A}{C}$ = B. *W.W. to be Dem.*

a hyp.
b 9. ax. 7.
c 7. ax. 7.

PROP. XXXX.

A, 15.
B, 3. C, 5.

If a number A *have any part whatsoever* B, *the number* C, *from which the part* B *is denominated, shall measure the same.*

For being BC [a] = A, [b] thence $\frac{A}{C}$ = B. *W.W. to be Dem.*

a hyp. and 9. ax. 7.
b 7. ax. 7.

PROP. XXXXI.

$\frac{1}{2}$ G, 12.
$\frac{1}{3}$ H - - -
$\frac{1}{4}$

To find out a number G, *which being the least, containeth the parts given* $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

[a] Let G be found the least which the denominators 2, 3, 4, measure; [b] it is evident that G ha's the parts $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. If it be possible let H ⊐ G have the same parts; [c] therefore 2, 3, 4, measure H; and so G is not the least which 2, 3, 4 measure: *against the constr.*

a 38. 7.
b 39. 7.
c 40. 7.

The End of the seventh Book.

THE EIGHTH BOOK OF EUCLIDE'S ELEMENTS.

Prop. I.

A, 8. B, 12. C, 18. D, 27.
E - F - - G - - - H - - - -

If there be divers numbers how many soever in continuall proportion, A, B, C, D, *and their extremes* A, D, *prime to one another; then those numbers* A, B, C, D, *are the least of all numbers that have the same proportion with them.*

For, if it be possible, let there be as many others E, F, G, H, lesse then A, B, C, D, and in the same proportion with them. a Therefore of equality A. D :: E. H. and consequently A & D are prime numbers, b and so the least in their proportion, c equally measuring E and H, which are lesse then themselves. *Which is Absurd.*

a 14. 7.
b 23. 7.
c 21. 7.

Prop. II.

1.
A, 2. B, 3.
Aq, 4. AB, 6. Bq, 9.
Ac, 8. AqB, 12. ABq, 18. Bc, 27.

To find out the least numbers continually proportionall, as many as shall be required, in the proportion given of A *to* B.

Let A and B be the least in the proportion given; Then Aq, AB, Bq, shall be the three least in the same continuall proportion that A is to B.

For

For AA. AB a :: A. B a :: AB. BB. Likewise because A and B are b prime to one another, c shall Aq, Bq, be also prime to one another, d and so Aq, AB, Bq, are ∺ the least in the proportion of A to B. a 17. 7. b 24. 7. c 29. 7. d 1. 8.

Moreover, I say Ac, AqB, ABq, Bc, are the four least in the proportion of A to B. For AqA. AqB e :: A.B. e :: ABA (AqB.) ABB. e and A.B :: ABq. BBq (Bc.) Therefore since Ac, and Bc, are f prime to one another, likewise g shall Ac, AqB, ABq, Bc be the four least ∺ in the proportion of A to B. In the same manner may you find out as many proportionall numbers as you please. *W. W. to be Done.* e 17. 7. f 29. 7. g 1. 8.

Coroll.

1. Hence, If three numbers, being the least, are proportionall, their extremes shall be squares; if four, cubes.

2. How many extremes proportionall soever there be, being by this propos. found to be the least in the given proportion, they are prime to one another.

3. Two numbers, being the least in the given proportion, doe measure all the mean numbers whatsoever of the least in the same proportion; because they arise from the multiplication of them into certain other numbers.

4. Hence also it appears by the construction, that the series of numbers 1, A, Aq, AC; 1, B, Bq, Bc; Ac, AqB, ABq, BC, consists of an equall multitude of numbers; and consequently, the extreme numbers of how many soever the least continually proportionals are the last of as many other continually proportionalls from a unite. as the extremes Ac, Bc, of the continually proportionals Ac, AqB, ABq, Bc, are the last of as many proportionals from a unite. 1, A, Aq, Ac; and 1, B, Bq, Bc.

5. 1, A, Aq, Ac; and B, BA, BAq; and Bq, ABq, are ∺ in the proportion of 1 to A. Also B, Bq, Bc; and A, AB, ABq; and Aq, AqB are ∺ in the proportion of 1 to B.

PROP. III.

A, 8. B, 12. C, 18. D, 28. *If there be numbers continually proportionall, how many soever,* A, B, C, D, *being also the least of all that have the same proportion with them; their extremes* A, D, *are prime to one another.*

a 2. 8. For if there be *a* found as many numbers the least in the proportion of A to B, they shall be no other then A, B, C, D; therefore, by the second *Coroll.* of the precedent *prop.* the extremes A and D are prime to one another. *W.W. to be Dem.*

PROP. IV.

A, 6. B, 5. C, 4. D, 3.
H, 4. F, 24. E, 20. G, 15.
I - - K - - L - - -

*Proportions how many soever being given in the least numbers (*A, *to* B, *and* C *to* D*) to find out the least numbers continually proportionall in the proportions given.*

a 36. 7. *a* Find out E the least number which B and C do
b 3. post 7. measure; and let B measure E *b* as often as A does an other F, *viz.* by the same number H. *b* Also let C measure the said E as often as D measures an other G. then F, E, G, shall be the least in the propor-
c 9 ax. 7. tions given. For AH *c* = F, and BC *c* = E; *d* there-
d 18 7. fore A.B :: AH. BH *e* :: F. E. In like manner C. D
e 7. 5. :: E.G; therefore F, E, G, are continually proportionall in the proportions given. And they are moreover the least in the said proportions: for con-
f 21. 7. ceive other numbers I, K, L, to be the least; *f* then A and B must equally measure I and K, *f* and C and D likewise K and L; and so B and C measure the same K. *g* Wherefore also E measures the same number
g 37. 7. K, which is lesse then it self. *Which is Absurd.*

A, 6.

A, 6. B, 5. C, 4. D, 3. E, 5. F, 7.
H, 24. G, 20. I, 15. K, 21.

But three proportions being given, A to B, C to D, and E to F; find out as before three numbers H, G, I, the least continually in the proportions of A to B, and C to D. Then if E measures I, [b] take another number K which may be equally measur'd by F; and those four numbers H, G, I, K, shall be continually the least in the given proportions; which we need go no other way to prove then we did in the first part. b 3. *pos*. 7.

A, 6. B, 5. C, 4. D, 3. E, 2. F, 7.
H, 24. G, 29. I, 25.
M, 48. L, 40. K, 30. N, 105.

If E doe not measure I, let K be the least which E and I doe measure; and as often as I measures K, let G as often measure L, and H also M. so likewise let F measure N as often as E measures K. The four numbers M, L, K, N shall be least continually in the given proportions; which we may demonstrate as before.

Prop. V.

C, 4. E, 3.
D, 6. F, 16. ED, 18.
$\overline{CD}$, 24. $\overline{EF}$, 48.

Plane numbers CD, EF, *are in that proportion to one another which is composed of their sides.*

For because $\overline{CD}.\overline{ED}$ [a] :: C. E; [a] and $\overline{ED}.\overline{EF}$ :: D. F. & $\frac{\overline{CD}}{\overline{EF}}$ [b] $= \frac{\overline{CD}}{\overline{ED}} + \frac{\overline{ED}}{\overline{EF}}$ [c] then shall be the proportion $\frac{\overline{CD}}{\overline{EF}} = \frac{C}{E} + \frac{D}{F}$. *W.W. to be Dem.*

a 17. 7.
b 20. *def*. 5.
c 11. 5.

Prop. VI.

A, 16. B, 24. C, 36. D, 54. E, 81.
F, 4. G, 6. H, 9.

If there be numbers continually proportionall how many soever, A, B, C, D, E, *and the first* A *doe not measure the second* B, *neither shall any of the other measure any one of the rest.*

a 10. def. 7. Because A does not measure B, [a] neither shall any one measure that which next followes; being A. B :: B.C :: C.D, &c. [b] Take three numbers, F, G, H the least in the proportion of A to B. therefore since A does not measure B, [a] neither shall F measure G. [c] therefore F is not a unite. But F and H are prime to one another; and so, [d] being of equality A.C :: F. H. and F does not measure H, [a] neither shall A measure C; and consequently neither shall B measure D, nor C measure E, &c. because A.C [e] :: B.D [e] :: C.E, &c. In like manner four or five numbers being taken the least in the proportion of A to B, it will appear that A does not measure D and E; nor does B measure E and F, &c. Wherefore none of them shall measure any other. *W. W. to be Dem.*

b 35. 7.

c 5. ax 7.

d 3. 3

e 14 7.

PROP. VII.

A, 3. B, 6. C, 12. D, 24. E, 48.

If there be numbers continually proportionall how many soever A, B, C, D, E, *and the first* A *measure the last* E, *it shall also measure the second* B.

a 6 7. If you deny that A measures B [a] then neither shall it measure E; *Which is contrary to the Hypoth.*

PROP. VIII.

A, 24.	C, 36.	D, 54.	B, 81.
G, 8.	H, 12.	I, 18.	K, 27.
E, 32.	L, 48.	M, 72.	F, 108.

If between two numbers A, B, *there fall mean proportionall numbers in continuall proportion* C, D; *as many mean continually proportionall numbers as fall between them, so many also mean continually proportionall numbers shall fall between two other numbers* E, F, *which have the same proportion with them.* (L, M.)

a 35 7. [a] Take G, H, I, K, the least ∺ in the proportion of A to C: [b] of equality shall G. K :: A. B [c] :: E. F. But G, and K [d] are prime to one another. [e] Wherefore G measures E as often as K does F. Let H measure L, and I likewise M by the same number. [f] therefore E, L, M, F are in such proportion as G, H, I, K, that is as A, B, C, D. *W. W. to be Dem.*

b 14 7

c hyp.

d 3. 8

e 21. 7.

f constr.

PROP.

Prop. IX.

1.
E,2. F,3.
G,4. H,6. I,9.
A,8. C,12. D,18. B,17.

If two numbers A, B *be prime to one another, and mean numbers in continual proportion* C, D *fall between them; as many mean numbers in continuall proportion as fall between them, so many means also*(E,G; and F,I) *shall fall in continuall proportion between either of them and a unitie.*

It is evident, that 1,E,G,A, and 1,F,I,B, are ∺, and as many as A,C,D,B. namely by the 4. Coroll. 2. 8. *W.W. to be Dem.*

Prop. X.

A,8. I,12. K,18. B,27.
E,4. DF,6. G,9.
D,2. F,3.
1.

If between two numbers A,B, *and a unite, numbers continually proportionall* (E,D, and F,G,) *do fall, how many mean numbers in continuall proportion fall between either of them and a unite, so many means also shall fall in continuall proportion between them,* I,K.

Nam E, DF, G, and A, DqF (I) DG (K) B are ∺ by 2.8. therefore, &c.

Prop. XI.

A,2. B,3.
Aq,4. AB,6. Bq,9.

Between two square numbers Aq, Bq, *there is one mean proportionall number* A B: *and the square* Aq *to the square* Bq *is in double proportion of that of the side* A *to the side* B.

a It is manifest that Aq, AB, Bq, are ∺; b and consequently also $\frac{Aq}{Bq}$ — $\frac{A}{B}$ doubly. *W.W. to be Dem.* a 17.7. b 10. *def.* 5.

Prop.

PROP. XII.

Ac,27. AqB, 36. ABq, 48. Bc, 64.
A, 3. B, 4.
Aq,9. AB, 12. Bq,16.

Between two cube numbers, Ac, Bc, *there are two mean proportionall numbers* AqB, ABq: *and the cube* Ac *is to the cube* Bc *in treble proportion of that in which the side* A *is to the side* B.

a 2. 8. b 10. def. 5. [a] For Ac,AqB,ABq,Bc, are ∺ in the proportion of A to B; [b] and therefore $\frac{Ac}{Bc} = \frac{A}{B}$ trebly. *W. W. to be Dem.*

PROP. XIII.

A, 2. B, 4. C, 8.
Aq,4. AB, 8. Bq, 16. BC,32. Cq,64.
Ac,8. AqB,16. ABq,32. Bc,64. BqC,128. BCq,256 Cc,512.

If there be numbers in continuall proportion how many soever A, B, C; *and every of them multiplying it self produce certain numbers; the numbers produced of them* Aq, Bq, Cq, *shall be proportionall: And if the numbers first given* A, B, C, *multiplying their products* Aq, Bq, Cq, *produce other numbers,* Ac, Bc, Cc, *they also shall be proportionall; and this shall ever happen to the extremes.*

a 2. 8. b 14. 7. For Aq,AB,Bq,BC,Cq [a] are ∺; [b] therefore of equality Aq.Bq :: Bq.Cq. *W.W. to be Dem.*

[a] Also Ac,AqB, ABq,Bc, BqC, BCq, Cc, are ∺; [b] therefore likewise of equality Ac. Bc :: Bc. Cc. *W.W. to be Dem.*

PROP. XIV.

Aq,4. AB,12. Bq,36.
A, 2. B, 6.

If a square number Aq *measure a square number* Bq, *the side also of the one* (A) *shall measure the side of the other* (B:) *& if the side of one square* A *measure the side of another* B, *the square* Aq *shall likewise measure the square* Bq.

1. *Hyp.*

1.*Hyp.* For Aq.AB a :: AB.Bq. therefore seeing by the hypothesis Aq measures Bq, b it shall measure also AB. But Aq. AB :: A. B. c therefore also A measures B. *W.W. to be Dem.* a 2. & 11. 8. b 7. 8. c 20. def. 7.

2.*Hyp.* A measures B. c therefore Aq shall as well measure AB, c as AB measures Bq; d and consequently Aq measures Bq. *W.W. to be Dem.* d 11. ax 7.

PROP. XV.

A, 2. B, 6.
Ac, 8. AqB, 24. ABq, 72. Bc, 216.

If a cube number Ac *measure a cube number* Bc, *then the side of the one* (A) *shall measure the side of the other* (B:) *And if the side* A *of one cube* Ac *measure the side* B *of the other* BC, *also the cube* Ac *shall measure the cube* Bc.

1.*Hyp.* For Ac, AqB, ABq, Bc a are ∺, therefore Ac, b measuring the extreme Bc, shall also c measure the second AqB. But Ac. AqB :: A. B. d therefore A shall also measure B. a 2. & 12 8. b hyp. c 7. 8. d 20. def. 7.

2.*Hyp.* A measures B; d therefore Ac measures AqB, which also measures ABq, and that Bc; e therefore Ac shall measure Bc. *W.W. to be Dem.* e 11. ax. 7.

PROP. XVI.

A, 4. B, 9.
Aq, 16. Bq, 81.

If a square number Aq *doe not measure a square number* Bq, *neither shall the side of the one* A *measure the side of the other* B: *And if* A *the side of the one square* Aq *doe not measure* B *the side of the other* Bq, *neither shall the square* Aq *measure the square* Bq.

1.*Hyp.* For if you affirm that A measures B, a then Aq also shall measure Bq. *against the Hyp.* a 14. 8.

2.*Hyp.* If you maintain Aq to measure Bq; a then likewise A shall measure B. *contrary to the Hypoth.*

Prop. XVII.

A, 2. B, 3.
Ac, 8. Bc, 27.

If a cube number Ac *doe not measure a cube number* Bc, *neither shall the side of one* A, *measure the side of the other* B: *And if* A *the side of one cube* Ac *do not measure* B *the side of the other* Bc, *neither shall the cube* Ac *measure the cube* BC.

a 15.8. 1. *Hyp.* Let A measure B; a then Ac shall measure Bc. *against the Hyp.*

2. *Hyp.* Let Ac measure Bc; then A shall measure B; *which is also against the Hyp.*

Prop. XVIII.

C, 6. D, 2.
CD, 12.
E, 9. F, 3. DE, 18.
EF, 27.

Between two like plane numbers C D *and* E F *there is one mean proportional number* DE: *And the plane* C D *is to the plane* EF *in double proportion of that which the side* C *hath to the homologous side (or of like proportion)* E.

* 21.def.7. a 17.7. b 11.5. c 10.def.5. Being * by the Hypoth. C.D :: E.F. therefore by inversion C.E :: D.F. But C.E a :: CD. DE; a and D.F :: DE. EF. b therefore CD. DE :: DE. EF. c Wherefore the proportion of CD to EF is double to that of CD to DE, that is, to the proportion of C to E, or D to F.

Coroll.

Hence it is apparent, That between two like plane numbers there falls one mean proportionall in the proportion of the homologous sides.

Prop.

PROP. XIX.

CDE, 30. DEF, 60. FGE, 120. FGH, 240.
CD, 6. DF, 12. FG, 24.
C, 2. D, 3. E, 5. F, 4. G, 6. H, 10.

Between two like solid numbers CDE, FGH, *there are two mean proportionall numbers* DFE, FGE. *And the solid* CDE *is to the solid* FGH, *in treble proportion of that which the homologous side* C *ha's to the homologous side* F.

Whereas by the * hyp. C.D :: F.G, & D.E :: G.H. therefore a by inversion shall C.F :: D. G :: a E. H. But CD.DF b :: C.F, & DF.FG b :: D.G; c wherefore CD.DF :: DF.FG :: E. H. d and accordingly CDE. DFE :: DFE.FGE :: E. H :: FGE.FGH. Therefore between CDE, FGH, fall two mean proportionals DFE, FGE. e And so it is plain that the proportion of CDE to FGH is treble to that of CDE to DFE, or C to F. *W.W. to be Dem.* * 21. def. 7. a 13. 7. b 17. 7. c 11. 5. d 17. 7. e 10. def. 5.

Coroll.

Hereby it is manifest, That between two like solid numbers there fall two mean proportionals in the proportion of the homologous sides.

PROP. XX.

A, 12, C, 18. B, 27.
D, 1. E, 3. F, 6. G, 9.

If between two numbers A, B, *there fall one mean proportionall number* C; *those numbers* A, B, *are like plane numbers.*

a Take D and E the least in the proportion of A to C, or C to B. then D measures A equally as E does C, *viz.* by the same number F; b also D equally measures C, as E does B *viz.* by the same number G. c Therefore DF = A, and EG = B. d and consequently A and B are plane numbers. But because EF c = C c = DG, e shall D.E :: F. G. and alternately a 35. 7. b 21. 7. c 9. ax. 7. d 16. def. 7. e 19. 7.

f 21.def.7. nately D.F :: E. G. *f* Therefore the plane numbers A and B are also like. *W.W. to be Dem.*

Prop. XXI.

A,16. C,24. D,36. B,54.
E,4. F,6. G,9.
H,2. P,2. M,4. K,3. L,3. N,6.

If between two numbers A, B *there fall two mean proportionall numbers* C, D; *those numbers* A, B *are like solid numbers.*

a 2.8. *a* Take E, F, G, the least ∺ in the proportion of
b 10.8. A to C. *b* then E and G are like plane numbers: let the sides of this be H & P, & of that K and L. *c* therefore
c 21.def.7. H.K :: P.L :: *d* E.F. But E,F,G, doe *e* equally mea-
d cor.18.8. sure A, C, D. *viz.* by the same number M. and like-
e 21.7. wise the said numbers E, F, G, doe equally measure the numbers C, D, B, *viz.* by the same number N. *f* Therefore A = EM = HPM, *f* and B = GN =
f 9.ax.7. KLN; *g* and so *g* A and B are solid numbers. But for
g 17 def.7. that C *f* = FM, and D *f* = FN, therefore shall M.N
h 17.7. *h* :: FM.FN *k* :: C.D *l* :: E.F :: H.K :: P.L. *m* where-
k 7.5.
l constr.
m 21.def.7. fore A and B are like solid numbers. *W.W. to be Dem.*

Lemma.

AE, BF, CG, DH,
A, B, C, D,
E, F, G, H.

If proportionall numbers A, B, C, D *measure proportionall numbers* AE, BF, CG, DH *by the numbers* E, F, G, H, *these numbers* (E, F, G, H) *shall be proportionall.*

a 19.7. For being AEDH *a* = BFCG, *a* and AD = BC,
b 1.ax.7. *b* thence will $\underbrace{A}E\underbrace{D}H = \underbrace{B}F\underbrace{C}G$ *c* that is, EH = FG.
c 9.ax.7. (AD ... BC,)

d Therefore E.F :: G.H. *W.W. to be Dem.*

Coroll.

Hence $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$. *d* For 1.B :: B.Bq *d* and 1.A ::
d 15.def.7.
e lem.prec. A.Aq. *e* therefore $\frac{1}{A}.\,B :: \frac{B.Bq}{AAq}$ *d* therefore $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$.
In like manner $\frac{B}{Ac} \times \frac{B}{Ac} = \frac{Bc}{Acc}$. and so of the rest.

Prop.

PROP. XXII.

Aq, B, C.
4, 8, 16.

If three numbers Aq, B, C *be continually proportionall, and the first* Aq *a square, the third* C *shall also be a square.*

For because AqC a= Bq, b thence is $C = \frac{Bq}{Aq}$ c= $Q.\frac{B}{A}$. But it is plain that $\frac{B}{A}$ is a number, d because $\frac{Bq}{Aq}$, or C is a number. Therefore if three, &c.

a 20. 7. b 7. ax. 7. c cor. of the lem. prec. d hyp. and 14. 8.

PROP. XXIII.

Ac, B, C, D.
8, 12, 18, 27.

If four numbers A, B, C, D *be continually proportionall, and the first of them* Ac *a cube, the fourth also* D *shall be a cube.*

For because AcD a= BC, b therefore $D = \frac{BC}{Ac}$ c= $\frac{B}{Ac} \times C$; that is (because AcC = d Bq, and b thence $C = \frac{Bq}{Ac}$) $D = \frac{B}{Ac} \times \frac{Bq}{Ac}$ c= $\frac{Bc}{Acc}$ c= $C : \frac{B}{Ac}$. But it is evident e that $\frac{B}{Ac}$ is a number, because $\frac{Bc}{Acc}$ or D is supposed a number. Therefore if four numbers, &c.

a 19. 7. b 7. ax. 1. c cor. of the prec. lem. d 20. 7. e 15. 8.

PROP. XXIV.

A,16. 24. B,36.
C,4. 6. D,9.

If two numbers A, B, *be in the same proportien one to another, that a square number* C *is to a square number* D, *and the first* A *be a square number, the second also* B *shall be a square number.*

Between C and D being square numbers, * and so between A and B having the same proportion, a falls one mean proportionall. Therefore b being A is

* 8. 8. a 11. 8. b hyp.

c 22. 8. is a square number, *c* B also shall be a square number. *W.W. to be Dem.*

Coroll.

1. Hence, If there be two like numbers AB, CD (A.B :: C.D) and the first AB be a square, the second also CD shall be a square.

* 12, & 18. 8. * For AB.CD :: Aq.Cq.

2. From hence it appears, That the proportion of any square number to any other not square cannot possibly be declared in two square numbers. Whence it cannot be Q.Q :: 1. 2. nor 1.5 :: Q.Q, &c.

P R O P. XXV.

C, 64. 96. 144. D, 216.
A, 8, 12. 18. B, 27.

If two numbers A, B, *be in the same proportion one to another, that a cube number* C *is to a cube number* D, *the first of them* A *being a cube number; the second* B *shall likewise be a cube number.*

a 12. 8. b 8. 8. c hyp. d 23. 8. *a* Between the cube numbers C and D, *b* and so between A and B having the same proportion, fall two mean proportionals; therefore *c* because A is a cube, *d* shall B be a cube also. *W.W. to be Dem.*

Coroll.

1. Hence, If there be two numbers ABC, DEF (A.B :: D.E, and B.C :: E. F;) and the first ABC

* 12, & 19. 8. be a cube, the second DEF shall be a cube also.

* For ABC.DEF :: Ac.Dc.

2. It is perspicuous from hence, That the proportion of any cube number to any other number not a cube cannot be found in two cube numbers.

PROP. XXVI.

A,20. C,30. B,45. *Like plane numbers* A, B,
D, 4. E, 6. F,9. *are in the ſame proportion one to another, that a ſquare number is in to a ſquare number.*

Between A and B *a* falls one mean proportionall number C ; *b* take three numbers D, E, F the leaſt ∺ in the proportion of A to C. the extremes D, F, *b* ſhall be ſquare numbers. But of equality A. B *c* :: D.F. therefore A.B :: Q. Q. *W.W. to be Dem.* a 18. 8. b 2. 8. c 14.7.

PROP. XXVII.

A,16. C,24. D,36. B,54. *Like ſolid numbers*
E,8. F,12. G,18. H,27. A, B, *are in the ſame proportion one to another, that a cube number is in to a cube number.*

a Between A and B fall two mean proportionall numbers, namely C and D : *b* take four numbers E, F, G, H the leaſt ∺ in the proportion of A to C; *b* the extremes E, H, are cube numbers. But A.B *c* :: E.H :: C.C. *W.W. to be Dem.* a 19. 8. b 2. 8. c 14.7.

Schol.

1. From hence is inferred, that no numbers in proportion ſuperparticular, ſuperbipartient, or double, or any other manifold proportion not denominated from a ſquare number, are like plane numbers. See Clavius.

2. Likewiſe, that neither any two prime numbers, nor any two numbers prime one to another, not being ſquares, can be like plane numbers.

The End of the eighth Book.

THE NINTH BOOK OF EUCLIDE'S ELEMENTS.

PROPOSITION I.

A, 6. B, 54.
Aq, 36. 108. AB, 324.

If two like plane numbers A, B *multiplying one another, produce a number* AB, *the number produced* AB *shall be a square number.*

a 17. 7. b 18. 8. c 8. 8. For A. B *a* :: Aq. AB; wherefore since one mean proportionall *b* falls between A and B, *c* likewise one mean proportionall number shall fall between Aq and AB: therefore being the first Aq is a square number, d 22. 8. *d* the third AB shall be a square number too. *W.W. to be Dem.*

x 19. 7. y 1. 2x. 7. Or thus. Let a b, c d, be like plane numbers; namely a. b :: c. d. *x* therefore a d = b c. and so likewise a b c d, or a d b c *y* = ad ad = Q: ad.

PROP. II.

A, 6. B, 54.
Aq, 36. AB, 324.

If two numbers A, B, *multiplying one another, produce a square number* AB, *those numbers* A, B *are like plane numbers.*

a 17. 7. b 11. 8. c 8. 8. d 20. 8. For A. B *a* :: Aq. AB; wherefore being between Aq, AB, *b* there falls one mean proportionall number, *c* likewise one mean shall fall between A and B. *d* therefore A and B are like planes. *W.W. to be Dem.*

PROP.

PROP. III.

A, 2. Ac, 8. Acc, 64. *If a cube number* Ac *multiplying it self produce a number* Acc, *the number produced* Acc *shall be a cube number.*

For 1. A a :: A. Aq b :: Aq. Ac. therefore between 1 and Ac fall two mean proportionalls. But 1. Ac a :: Ac. Acc. c therefore between Ac and Acc, fall also two mean proportionalls : and so by consequence seeing Ac is a cube, d Acc shall be a cube also. *Which was to be Dem.*

a 15. def. 7.
b 17. 7.
c 8. 8.
d 23 8.

Or thus; aaa (Ac) multiplyed into it self makes aaaaaa (Acc;) this is a cube, whose side is aa.

PROP. IV.

Ac, 8. Bc, 27, Acc, 64. AcBc, 216. *If a cube number* Ac *multiplying a cube number* Bc *produce a number* AcBc, *the produced number* AcBc *shall be a cube.*

For Ac. Bc a :: Acc. AcBc. But between Ac and Bc b two mean proportionall numbers fall ; c therefore there fall as many between Acc and AcBc. So that whereas Acc is a cube number, d AcBc shall be such also. *W.W. to be Dem.*

a 17. 7.
b 12. 8.
c 8. 8.
d 23 8.

Or thus. AcBc = aaabbb (ababab) = C: ab.

PROP. V.

Ac, 8. B, 27. Acc, 64. AcB, 216. *If a cube number* Ac *multiplying a number* B *produce a cube number* AcB, *the number multiplyed* B *shall also be a cube.*

For Acc. AcB a :: Ac. B. But between Acc and AcB b fall two mean proportionalls; c therefore also as many shall fall between Ac and B. whence Ac being a cube number, d B shall be a cube number too. *W.W. to be Dem.*

a 17. 7.
b 12. 8.
c 8. 8.
d 23 8.

PROP. VI.

A, 8. Aq, 64. Ac, 512. *If a number* A *multiplying it self produce a cube* Aq, *that number* A *it self is a cube.*

a hyp. b 19. def. 7. c 5. 9. For because Aq *a* is a cube, and AqA (Ac) *b* also a cube; therefore *c* shall A be a cube. *W.W. to be Dem.*

PROP. VII.

A, 6. B, 11. AB, 66. D, 2. E, 3. *If a composed number* A *multiplying any number* B, *produce a number* AB, *the number produced* AB *shall be a solid number.*

a 13. def. 7. b 9. ax. 7. c 17. def. 7. Being A is a composed number, *a* some other number D measures it, conceive by E. *b* therefore A = DE: *c* whence DEB = AB is a solid number. *W.W. to be Dem.*

PROP. VIII.

1. a, 3. a2, 9. a3, 27. a4, 81. a5, 243. a6, 729.

If from a unite there be numbers continually proportionall how many soever (1, a, a2, a3, a4, &c.) *the third number from a unite* a2 *is a square number; and so are all forward, leaving one between* (a4, a6, a8, &c.) *But the fourth* a3 *is a cube number; and so are all forward, leaving two between* (a6, a9, &c.) *The seventh also* a6 *is both a cube number and a square; and so are all forward, leaving five between* (a12, a18, &c.)

For 1. a2 = Q.a. and a4 = aaaa = Q. aa. and a6 = aaaaaa = Q. aaa, &c.

2. a3 = aaa = C.a. and a6 = aaaaaa = C.aa. and aaaaaaaaa = C.aaa, &c.

3. a6 = aaaaaa = C.aa = Q.aaa. therefore, &c.

a hyp. b 20. 7. c 12. 8. Or according to *Euclide*; Because 1.a *a* :: a. a2. *b* shall a2 = Q:a. therefore seeing a2, a3, a4, are ∺, *c* the third a4 shall be a square number; and so likewise a6, a8, &c. Also because 1. a *a* :: a2. a3. therefore

d 23. 8. shall a3 *b* = a2 x a = C: a. *d* therefore the fourth from a3, namely a6, shall be likewise a cube, &c. and consequently a6 is both a cube and a square number, &c.

PROP. IX.

1. a, 4. a^2, 16. a^3, 64. a^4, 256, &c.
1. a, 8. a^2, 64. a^3, 512. a^4, 4096.

If from a unite there be numbers how many soever continually proportionall (1, a, a^2, a^3, &c.) *and the number following the unite* (a) *be a square; then all the rest,* a^2, a^3, a^4, &c. *shall be squares too. But if the number next the unite* (a) *be a cube, then all the following numbers* a^2, a^3, a^4, &c *shall be cube numbers.*

1. *Hyp.* For a^2, a^4, a^6, &c. are square numbers by the prec. *prop.* also being a is taken to be a square, a therefore the third a^3 shall be a square, and likewise a^5, a^7, &c. and so all. a 22. 8.

2. *Hyp.* a is taken to be a cube, b therefore a^4, a^7, a^{10} are cubes: but by the prec. a^3, a^6, a^9, &c. are cubes: lastly because 1. a :: a. aa. c therefore shall a^2 = Q: a. but a cube multiplyed into it self d produces a cube; therefore a^2 is a cube, e and consequently the fourth from it a^5, and in like manner a^8, a^{11}, &c. are cubes. therefore all. *W. W. to be Dem.* b 23. 8. c 10. 7. d 3. 9. e 23. 8.

Peradventure more clearly thus. Let b be the side of the square number a, and so the series a, a^2, a^3, a^4, &c. will be otherwise expressed, thus, bb, b^4, b^6, b^8, &c. It is evident that all these numbers are squares, and may be thus expressed, Q: b, Q: bb: Q. bbb, Q: bbbb, &c.

In like manner, if b be the side of the cube a, the series may be expressed thus, b^3, b^6, b^9, b^{12}, &c. or C: b, C: b^2, C: b^3, C: b^4, &c.

PROP. X.

1, a, a^2, a^3, a^4, a^5, a^6.
1, 2, 4, 8, 16, 32, 64.

If from a unite there be numbers how many soever continually proportionall (1, a, a^2, a^3, &c.) *and the number next the unite* (a) *be not a square number; then is none of the rest following a square number, excepting* a^2 *the third from the unite, and so all forward, leaving one between* (a^4, a^6, a^8, &c.)

But if that (a) which is next after the unite, be not a cube number, neither is any other of the following numbers a cube, saving a^3 the fourth from the unite, and so all forward, leaving two between, a^6, a^9, a^{12}, &c.

a hyp. b suppos and 8. 9. c 24. 8. 1. *Hyp.* For it it be possible, let a^5 be a square number; therefore because $a. a^2$ a :: $a^4. a^5$, and by inversion $a^5. a^4$ :: $a^2. a$; and also a^5 and a^4 b square numbers, and the first a^2 a square, c therefore a shall be likewise a square; *contrary to the Hyp.*

d 24. 7. 2. *Hyp.* If it may be, let a^4 be a cube; being d of equality $a^4. a^6$ :: $a. a^3$, and inversely $a^6. a^4$ :: $a^3. a$; and also being a^6 and a^4 are cubes, and the first a^3 a cube, e therefore a shall be a cube also; *against the Hyp.*

e 25. 8.

PROP. XI.

$1. a, a^2, a^3, a^4, a^5, a^6.$
$1, 3, 9, 27, 81, 243, 729.$

If there be numbers how many soever in continuall proportion from a unite (1, a, a^2, a^3, &c.) the lesse measureth the greater by some one of them that are amongst the proportionall numbers.

a 5. ax. 7. & 20 def. 7. b 14. 7. Because 1. a :: a. aa. a therefore $\frac{aa}{a} = a = \frac{aaa}{aa}$. Also because 1. aa b :: a. aaa. a therefore $\frac{aaa}{a} = aa = \frac{a^4}{aa} = \frac{a^5}{a^3}$, &c. Lastly because 1. a^3 b :: a. a^4. therefore $\frac{a^4}{a} = a^3 = \frac{a^6}{a^3}$, &c.

Coroll.

Hence, If a number that measures any one of proportional numbers, be not one of the said numbers, neither shall the number by which it measures the said proportionall numbers, be one of them.

PROP.

PROP. XII.

1, a, a^2, a^3, a^4.
1, 6, 36, 216, 1296.
B, 3.

If there be numbers how many ſoever in continuall proportion from a unite (1, a, a^2, a^3, a^4.) *whatſoever prime numbers* B *meaſure the laſt* a^4, *the ſame* (B) *ſhall alſo meaſure the number* (a) *which follows next after the unite.*

If you ſay B does not meaſure a, *a* then B is prime to a; *b* and alſo B is prime to a^2; *c* and ſo conſequently to a^4, which it is ſuppoſed to meaſure. *Which is Abſurd.* a 31. 7. b 27. 7. c 28. 7.

Coroll.

1. Therefore every prime number that meaſures the laſt, does alſo meaſure all thoſe other numbers that precede the laſt.

2. If any number not meaſuring that next to the unite, does yet meaſure the laſt, it is a compoſed number.

3. If the number next to the unite be a prime, no other prime number ſhall meaſure the laſt.

PROP. XIII.

1, a. a^2, a^3, a^4,
1, 5, 25, 125. 625.
H - - G - - F - - E - -

If from a unite be numbers in continuall proportion how many ſoever (a, a^2, a^3, &c.) *and that after the unite* (a) *a prime; then ſhall no other meaſure the greateſt number, but thoſe which are amongſt the ſaid proportionall numbers.*

If it be poſſible, let ſome other E meaſure a^4, *viz.* by F. *a* then F ſhall be ſome other beſide a, a^2, a^3. But becauſe E meaſuring a^4, does not meaſure a, *b* therefore E ſhall be a compoſed number, *c* and ſome prime number meaſure it, *d* which does conſequently meaſure a^4. *e* and ſo is no other then a. therefore a meaſures E. So alſo may F be ſhewn to be a com- a cor. 11. 9. b 2. cor. 12. 9. c 33. 7. d 11. ax. 7. e 3. cor. 12. 9.

 poſed

posed number, measuring a^4, and so that a measures
f 9. ax. 7. F. Therefore seeing EF f $= a^4 = a \times a^3$, g shall a.
g 19. 7. E :: F. a^3. Consequently, whereas a measures E,
h 20. def. 7. h likewise F shall equally measure a^3, *viz.* by the
k cor. 11. 9. same number G. k Nor shall G be a, or a^2. therefore, as before, G is a composed number, and a measures it. Wherefore being that FG f $= a^3 = a^2 \times a$ g shall a.F :: G. a^2. and so because A measures F, h G shall equally measure a^2. *viz.* by the same number H, k which is not a. Therefore being GH $= a^2 =$ aa,
l 23. def. 7. l thence H. a :: a.G. & because a measures G (as be-
m 20. def. 7. fore) m H also shall measure a, which is a prime number. *Which is impossible.*

PROP. XIV.

A,30.
B,2. C,3. D,5.
E-- F---

If certain prime numbers B, C, D, *do measure the least number* A, *no other prime number* E *shall measure the same, besides those that measured it at first.*

a 9. ax 7. If it is possible, let $\frac{A}{E}$ be $=$ F. a then A $=$ EF.
b 32. 7. b therefore every of the prime numbers B, C, D measures one of those E, F. not E, which is taken to be a prime; therefore F, which is lesse then it self A; *contrary to the Hyp.*

PROP. XV.

A, 9. B, 12, C, 16.
D, 3. E, 4.

If three numbers continually proportionall A, B, C, *be the least of all that have the same proportion with them; any two of them added together shall be prime to the third.*

a 35. 7. a Take D and E the least in the proportion of A
b 2. 8. to B; b then A $=$ Dq, & b C$=$Eq, b and B $=$ DE.

But

But because D [c] is prime to E, [d] therefore shall D + E be prime to both D and E. * therefore D x D + E, = Dq + DE (*f* A + B) is prime to E, and so to C or Eq. *W.W. to be Dem.* c 14.7. d 30.7. * 26.7. e 3.2. f before.

[g] In like manner DE + Eq (B + C) is prime to D, and consequently to A = Dq. *W.W. to be Dem.* g 27.7.

Lastly, because B [h] is prime to D + E, it shall also be prime to the square of it [k] Dq + 2 DE + Eq (A + 2 B + C;) [l] wherefore the said B shall be prime to A + B + C, [l] and so likewise to A + C. *Which was to be Dem.* h 26.7. k 4.2. l 30.7.

PROP. XVI.

A, 3. B, 5. C - - -

If two numbers A, B, *be prime to one another, the second* B *shall not be to any other* C, *as the first* A *is to the second* B.

If you affirm A. B :: B. C. then whereas A and B [a] are the least in their proportion, [b] shall measure B as many times as B does C; but A [c] measures it self also; therefore A and B are not prime to one another. *against the Hyp.* a 23.7. b 21.7. c 6. ax. 7.

PROP. XVII.

A, 8. B, 12. C, 18, D, 27. E - - -

If there be numbers how many soever in continuall proportion A, B, C, D, *and the extremes of them* A, D *be prime one to another, the last* D *shall not be to any other number* E, *as the first* A *is to the second* B.

Suppose A. B :: D. E. then alternately A. D :: B. E. therefore seeing A and B are [a] the least in their proportion, A [b] shall measure B, [c] and B likewise C, and C the following number D. [d] and so A shall measure the said number D. Wherefore A and a 23.7. b 21.7. c 22. d. f. 7. d 11. ax. 7.

D

D are not prime to one another , *contrary to the Hypoth.*

PROP. XVIII.

A, 4. B, 6. C, 9.
Bq, 36.

Two numbers being given A, B, *to consider if there may be a third number found proportionall to them* C.

a 9. ax. 7. b 10. 7. If A measure Bq by any number C, *a* then AC = Bq. from whence *b* it is manifest that A.B :: B.C. *W.W. to be Dem.*

A, 6. B, 4. Bq, 16.

But if A doe not measure Bq, there will not be any third proportionall. For suppose A.B :: B.C. *a* then AC = Bq.

c 7. ax. 7. *c* and consequently $\frac{Bq}{A}$ = C. namely A measures Bq. *Which is against the Hypoth.*

PROP. XIX.

A, 8. B, 12. C, 18. D, 27.
BC, 216.

Three numbers being given A, B, C, *to consider if a fourth proportionall to them* D *may be found.*

a 9. ax. 7. b ax. 19. 7. If A measures BC by any number D, *a* then AD = BC; *b* therefore it appears that A. B :: C. D. which was required.

But if A do not measure BC, then there can no fourth proportionall be found; which may be shewn as in the prec. *prop.*

PROP. XX.

A, 2. B, 3. C, 5.
D, 30. G - - - -

More prime numbers may be given then any multitude whatsoever of prime numbers A, B, C, *propounded.*

a 38. 7. b 33. 7. *a* Let D be the least which A, B, C, measure; If D + 1 be a prime, the case is plain; if composed, *b* then some prime number, conceive G, measures D + 1;

which

which is none of the three A, B, C; For if it be, seeing it [c] measures the whole D + 1, [d] and the part taken away D, [e] it shall also measure the remaining unite. *which is Abs.* Therefore the propounded number of prime numbers is increased by D + 1, or at least by G. c *suppos.* d *constr.* e 12. *ax.* 7.

PROP. XXI.

5 . 5 3 3 2 2

A..... E..... B... F... C.. G.. D 20.

If even numbers, how many soever, AB, BC, CD, *be added together, the whole* AD *shall be even.*

[a] Take EB $= \frac{1}{2}$ AB, and FC $= \frac{1}{2}$ BC, and GD $= \frac{1}{2}$ CD. [b] it is plain that EB + FC + GD $= \frac{1}{2}$ AD. [c] therefore AD is an even number. *Which was to be Dem.* a 6. *def.* 7. b 12. 7. c 6. *def.* 7.

PROP. XXII.

I I I

A....... F . B ...... G . C .... H . D .. L . E 22.

9 7 5 3

If odd numbers, how many soever, AB, BC, CD, DE, *be added together, and their multitudes even, the whole also* AE *shall be even.*

A unite being taken from each odde number, there will [a] remain AF, BG, CH, DL, even numbers, [b] and thence the number compounded of them will be even. adde to them the [c] even number made of the remaining unites, and the [d] whole AE will thereby be even. *W. W. to be Dem.* a 7. *def.* 7. b 21. 9. c *hyp.* d 21. 9.

PROP.

PROP. XXIII.

7 5 1
A .... B ..... C .. E . D 15.
3

If odd numbers how many soever AB, BC, CD, *be added together, and the multitude of them be odde, the whole* AD *shall be odde.*

a 22. 9.
b 21. 9.
c 7. def. 7.

For CD one of the odde numbers being taken away, the number compounded of the others AC a is even. Whereto adde CD − 1, b the whole AE is also even; wherefore the unite being restored the whole AD c will be odd. *W. W. to be Dem.*

PROP. XXIV.

4 5 1
A .... B ..... D . C 10.
6

If an even number AB *be taken away from an even number* AC, *that which remains* BC *shall be even.*

a 7. def. 7.
b hyp.
c 21. 9.

For if BD (BC − 1) be odde, a BC (BD + 1) will be even. *W. W. to be Dem.* But if you say BD is even, because AB b is even, c thence AD will be so; a & consequently AC (AD + 1) will be odde, *contrary to the Hypoth.* therefore BC is even. *W. W. to be Dem.*

PROP. XXV.

6 1 3
A ...... D . C ... B 10.
7

If from an even number AB, *an odde number* AC *be taken away, the remaining number* CB *shall be odde.*

a 7. def. 7.
b 24. 9.
c 7. def. 7.

For AC − 1 (AD) a is even. b therefore DB is even; c and consequently CB (DB − 1) is odd. *W. W. to be Dem.*

PROP. XXVI.

4 6 1
A ...., C ...... D . B 11.

If from an odde number AB *be taken away an odde number* CB, *that which remaineth* AC *shall be even.*

For

For AB − 1 (AD) and CB − 1 (CD) [a] are even; [b] therefore AD − CD (AC) is even. *W.W. to be Dem.* a 7. def. 7. b 24. 9.

Prop. XXVII.

1 4 6
A . D .... C ...... B 11
5

If from an odde number AB *be taken away an even number* CB, *the residue* AC *shall be odde.*

For AB − 1 (DB) [a] is even, and CB is supposed to be even; [b] therefore the residue CD is even: [c] therefore CD + 1 (CA) is odde. *W.W. to be Dem.* a 7. def. 7. b 24. 9. c 7. def. 7.

Prop. XXVIII.

A, 3
B, 4
AB, 12.

If an odde number A *multiplying an even number* B *produce a number* AB, *the number produced* AB *shall be even.*

For AB [a] is compounded of the odde number A taken as many times as a unite is conteined in B an even number. [b] Therefore AB is an even number. a hyp. and 15. def. 7. b 21. 9.

Schol.

In like manner, if A be an even number, AB shall be an even number also.

Prop. XXIX.

A, 3.
B, 5.
AB, 15.

If an odde number A *multiplying an odde number* B, *produce a number* AB, *the number produced* AB *shall be odde.*

For AB [a] is compounded of the odde number B taken as often as a unite is included in A likewise an odde number. [b] Therefore AB is an odd number. *W.W. to be Dem.* a 15. def. 7. b 23. 9.

Schol.

Schol.

B,12 (C,4.
A, 3

1. *An odd number* A *measuring an even number* B, *measures the same by an even number* C.

a 9 ax.7. b 29.9. For if C be affirmed to be odde, then because *a* B = AC, *b* therefore B shall be odde, *against the Hyp.*

B, 15 (C, 5
A, 3

2. *An odde number* A *measuring an odde number* B, *measures the same by an odde number* C.

a 28,9. For if C be said to be even, *a* then AC, or B will be even, *contrary to the Hypoth.*

B,15. (C,5.
A, 3.

3. *Every number* (A *and* C) *that measures an odde number* B, *is it self an odde number.*

a 28,9. For if either A or C be affirmed to be even, B *a* shall be an even number, *against the Hypoth.*

Prop. XXX.

B, 24. (C, 8. D, 12. (E,4.
A, 3 A, 3

If an odde number A *measure an even number* B, *it shall also measure the half of it* D.

a hyp. b 1.schol. 29.9. c 9 ax.7. d 1.2 e hyp. f 7.ax.1. g 7.ax.7.

a Let $\frac{B}{A}$ be = C. *b* then C is an even number. Therefore let E be = $\frac{1}{2}$ C, then B *c* = CA *d* = 2 EA *e* = 2 D. *f* therefore EA = D; *g* and consequently $\frac{D}{A}$ = E. *W.W. to be Dem.*

Prop. XXXI.

A,5. B,8. C,16. D ---

If an odde number A *be prime to any number* B, *it shall also be prime to the double thereof* C.

a 3.schol.29.9. b 30.9.

If it be possible, let some number D measure A and C, *a* then D measuring the odde number A shall be odde it self, *b* & so shall measure B the half of the even number C. therefore A & B are not prime one to another. *Which is against the Hyp.*

Coroll.

Coroll.

It follows from hence that an odde number which is prime to any number of double progression, is also prime to all the numbers of that progression.

PROP. XXXII.

1. A,2. B,4. C,8. D,16. *All numbers* A, B, C, D, &c. *in double progression from the binarie are evenly even only.*

It is evident that all these numbers 1, A, B, C, D, *a* are even, and *b* ∺, namely in a double proportion, *c* and so every lesse measures the greater by some one of them. *d* Wherefore all are evenly even. But for that A is a prime number, *e* no number beside these shall measure any of them. Therefore they are evenly even only. *W.W. to be Dem.*

a 6 def. 7.
b 20 def. 7.
c 12. 9.
d 8 def. 7.
e 13. 9.

PROP. XXXIII.

A,30. B,15.
D--- E--

If of a number A, *the half* B *be odd, the same* A *is evenly odd only.*

Being an odde number B *a* measures A by two an even number, *b* therefore B is evenly odde. If you affirm it to be evenly even, *c* then some even number D measures it by an even number E. whence 2 B *d* = A *d* = DE. *e* wherefore 2, E :: D. B. and therefore as 2 *f* measures the even number E, *g* so D an even number measures B an odde. *Which is impossible.*

a hyp.
b 9 def. 7.
c 8. def. 7.
d 9 ax. 7.
e 19 7.
f 6 def. 7.
g 10 def. 7.

PROP. XXXIV.

A, 24. *If an even number* A *be neither doubled from two, nor have it's half part odde, it is both evenly even and evenly odde.*

It is undoubtable, that A is evenly even, because the half of it is not odde. But because, if A be divided into two equall parts, and so continuing the bipartition

a 7.def.7. tition, we shall at length light upon some a odde number (not upon the number two, because A is not supposed to be doubled upward from two) which shall measure A by an even number. (for b o-
b 1.sch.29.9. therwise A it self should be odde, *against the Hyp.*) Therefore A is also evenly odde. *W.W. to be Dem.*

PROP. XXXV.

A .... 8.
4 8
B .... F .... G 12.
C .... 18.
9 6 4 8
D .... H ...... L .... K .... N 27.

If there be numbers in continuall proportion how many soever A - BG, C, DN, *and the number* FG *be taken from the second, and* KN *from the last, equall to the first* A; *as the excesse of the second* BF *is to the first* A, *so shall the excesse of the last* DK *be to all the numbers that precede it,* A, BG, C.

From DN take NL = BG, and NH = C. Be-
a hyp. cause DN. C (HN) a :: HN. BG (LN) a :: LN
b 17. 5. (BG.) A (KN.) b therefore by dividing each, shall
c 12. 5. DH.HN :: HL.LN :: LK. KN. c wherefore DK. C
d 3.ax.1. + BG + A :: LK (d BF.) KN (A.) *W. W. to be Dem.*

Coroll.

e 18. 5. Hence e by compounding, DN + BG + C. A + BG + C :: BG. A.

PROP. XXXVI.

1. A, 2. B, 4. C, 8. D, 16.
E, 31. G, 62. H, 124. L, 248. F, 496.
M, 31. N, 465.
P - - - Q - - -

If from a unite be taken how many numbers soever 1, A, B, C, D, *in double proportion continually, untill the whole added together* E *be a prime number; and if this whole*

whole E *multiplying the last produce a number* F, *that which is produced* F *shall be a perfect number.*

Take as many numbers E, G, H, L, likewise in double proportion continually; then [a] of equality A.D :: E.L. [b] therefore AL = DE [c] = F. [d] whence L = F. Wherefore E, G, H, L, F, are ∺ in double proportion. Let G − E be = M, and F − E = N; [e] then M.E :: N.E + G + H + L. [f] But M = E. [g] therefore N = E + G + H + L. [h] therefore F = 1 + B + C + D + E + G + H + L = E + N. Moreover because D [k] measures DE(F) [l] therefore every one, 1, A, B, C, [m] measuring D, as [m] also E, G, H, L, does measure F. And further, no other number measures the said F. For if there do, let it be P, which measures F by Q. [n] therefore PQ = F = DF. [o] therefore E.Q :: P.D. therefore seeing A a prime number measures D, [p] and so no other P measures the same, [q] consequently E does not measure Q. Wherefore E being supposed a prime number, [r] it shall be prime to Q. [s] wherefore E and Q are the least in their proportion; [t] and so E measures P as many times as Q does D; [u] therefore Q is one of them A, B, C. Let it be B. seeing then of equality B.D :: E.H. [x] and so BH = DE = F = PQ. [x] and so also Q.B :: H.P. [y] therefore H = P. therefore P is also one of them A, B, C, &c. *against the Hypoth.* Wherefore no other beside the foresaid numbers measures F, and [z] consequently F is a perfect number. *Which was to be Demonstrated.*

a 14. 7.
b 19. 7.
c hyp.
d 7. ax. 7.
e 35. 9.
f 3. ax. 1.
g 14. 5.
h 2. ax. 1.
k 7. ax. 7.
l 11. ax. 7.
m 11. 9.
n 9. ax. 7.
o 19. 7.
p 13. 9.
q 20. def. 7.
r 31. 7.
s 23. 7.
t 21. 7.
u 13. 7.
x 19. 7.
y 14. 5.
z 22. def. 7.

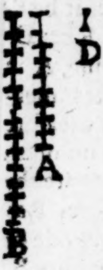
The End of the ninth Book.

N THE

THE TENTH BOOK OF EUCLIDE'S ELEMENTS.

Definitions.

I. Commenſurable magnitudes are thoſe, which are meaſured by one and the ſame meaſure.

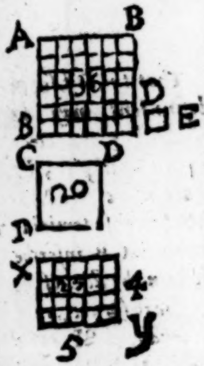


The note of commenſurability is ⊓⊔, *as* A ⊓⊔ B; *that is the line* A *of* 8 *foot is commenſurable to the line* B *of* 13 *foot; becauſe* D *a line of one foot meaſures both* A *&* B. *Alſo* $\surd 18$ ⊓⊔ $\surd 50$; *becauſe* $\surd 2$ *meaſures both* $\surd 18$ *and* $\surd 50$. *For* $\surd \frac{18}{2} = \surd 9 = 3$. *and* $\surd \frac{50}{2} = \surd 25 = 5$. *wherefore* $\surd 18 . \surd 50 :: 3 . 5$.

II. Incommenſurable magnitudes are ſuch, of which no common meaſure can be found.

Incommenſurability is denoted by this mark ⊓⊔; *as* $\surd 6$ ⊓⊔ $\surd 25$ (5;) *that is,* $\surd 6$ *is incommenſurable to the number* 5, *or to a magnitude deſigned by that number; becauſe there is no common meaſure of them, as ſhall appear hereafter.*

III. Right lines are commenſurable in power, when the ſame ſpace does meaſure their ſquares.



The mark of this commensurability is ⊐; *as* AB ⊐ CD. *i.e. the line* AB *of 6 foot is in power commensurable to the line* CD, *which is expressed by* √ 20. *because* E *the space of one foot square does as well measure* ABq (36) *as the rectangle* XY (20) *to which the square of the line* CD (√ 20) *is equall. The same note* ⊐ *sometimes signifies commensurable in power only.*

IV. Lines incommensurable in power are such, to whose squares no space can be found to be a common measure.

This incommensurability is denoted thus; 5 ⊐ ᵘ√ 8: *i.e. the numbers or lines* 5, *and* ᵘ√ 8 *are incommensurable in power, because their squares* 25 *and* √ 8 *are incommensurable.*

V. From which it is manifest, that to any right line given right lines infinite in multitude are both commensurable and incommensurable; some in length and power, others in power only. The right line given is called a Rationall line.

The note of which is ρ́.

VI. And lines commensurable to this line, whether in length and power, or in power only, are also called Rationall, ρ́.

VII. But such as are incommensurable to it, are called Irrationall,

And denoted thus ρ̀.

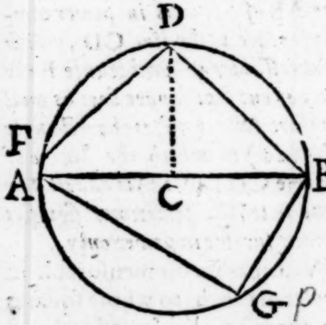
VIII. Also the square which is made of the said given right line is called Rationall, ρ́ν.

IX. And likewise such figures as are commensurable to it, are Rationall ρ́α.

X. But such as are incommensurable, Irrationall ρ̀α.

XI. And those right lines also, which contain them in power, are Irrationall ρ̀.

Schol.



That the last seven definitions may be rendred more clear by an example, let there be a circle ADBP, *whose semidiameter is* CB *inscribe therein the sides of the ordinate figures, as of a Hexagone* BP, *of a Triangle* AP, *of a square* BD, *of a Pentagone* FD. *Therefore, if according to the* 5. def. *the semidiameter* CB *be the Rationall line given, expressed by the number* 2, *to which the other lines* BP, AP, BD, FD, *are to be compared, then* P[a] = BC = 2. *wherefore* BP *is* ρ́ ⊐⊏ BC, *according to the* 6. def. *Also* AP[b] = √ 12 (*for* ABq (16) − BPq (4) = 12) *therefore* AB *is* ρ́ ⊐⊏ BC *likewise accord. to the* 6. def. *and* APq (12) *is* ρ́v *by the* 9. def. *Moreover* BD[b] = √ DCq + BCq = √ 8; *whence* BD *is* ρ́ ⊐⊏ BC; *and* BDq ρ́v *Lastly*, FDq = 10 − √ 20 (*as shall appear by the praxis to be delivered at the* 10. 13.) *shall be* ρ́v, *according to the* 10. def. & *consequently* FD = √ : 10 − √ 20 *is* ρ́, *according to the* 11. def.

a cor. 15. 4.
b 47. 1.

A Postulate.

That any magnitude may be so often multiplied, till it exceed any magnitude whatsoever of the same kind.

Axiomes.

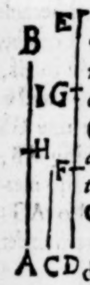
1. A magnitude measuring how many magnitudes soever, does also measure that which is composed of them.

2. A

2. A magnitude measuring any magnitude whatsoever, does likewise measure every magnitude which that measures.

3. A magnitude measuring a whole magnitude and a part of it taken away, does also measure the residue.

Prop. I.

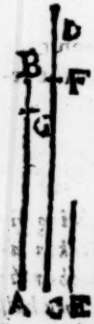


Two unequall magnitudes AB, C, *being given, if from the greater* AB *there be taken away more then half* (AH) *and from the residue* (HB) *be again taken away more then half* (HI) *and this be done continually, there shall at length be left a certain magnitude* IB, *lesse then the lesse of the magnitudes first given* C.

a Take C so often, till it's multiplex DE doe somewhat exceed AB, and there be DF = FG = GE = C. Take from AB more then half HA, and from the remainder HB more then half HI, and so continually, till the parts AH, HI, IB, be equall in multitude to the parts DF, FG, GE. Now it is plain, that FE, which is not lesse then $\frac{1}{2}$ DE, is greater then HB, which is lesse then $\frac{1}{2}$ AB $\sqsupset$ DE. And in like manner GE which is not lesse then $\frac{1}{2}$ FE, is greater then IB $\sqsupset$ $\frac{1}{2}$ HB. therefore C, or GE $\sqsubset$ IB. *W. W. to be Dem.* a post. 10.

The same may also be demonstrated, if from AB the half AH be taken away, and again from the residue HB the half HI, and so forward.

PROP. II.

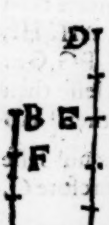


Two unequall magnitudes being given (AB,CD) if the lesse AB be continually taken from the greater CD, by an interchangeable substraction, and the residue doe not measure the magnitude going before, then are the magnitudes given incommensurable.

If it be possible, let some magnitude E be the common measure. Then because AB taken from CD, as often as it can be, leaves a magnitude FD lesse then it self, and FD taken from AB leaves GB, and so forward; [a] therefore at length some magnitude GB ⊐ E shall be left. therefore E [b] measuring AB, [c] and so CF, [b] and the whole CD, [d] shall also measure the residue FD. [c] consequently also AG; [d] wherefore it shall likewise measure the remainder GB, lesse then it self. *Which is Absurd.*

a 1.10.
b hyp.
c 2.ax.10.
d 3.ax.10.

PROP. III.



Two commensurable magnitudes being given AB, CD, to find out their greatest common measure FB.

Take AB from CD, and the residue ED from AB, and FB from ED, till FB measure ED (which will come to passe at length, [a] because by the Hyp. AB ⊏ CD) FB shall be the magnitude required.

a 2.10.

For FB [b] measures ED, [c] and so also AF; but it measures it self too, [d] therefore likewise AB, [e] and consequently CE, [d] and so the whole CD. Wherefore FB is the common measure of AB, CD. If you affirm G to be a common measure greater then that, then G measuring AB and CD, [e] measures also CE and [f] the remainder ED, [e] and so AF; and [f] consequently the remainder FB, the greater the lesse. *Which is Absurd.*

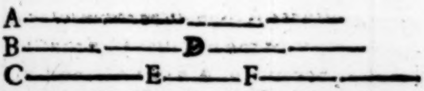
b constr.
c 2.ax.10.
d 1.ax.10.
e 2.ax.10.
f 3.ax.10.

Co-

Coroll.

Hence, A magnitude that measures two magnitudes, does also measure their greatest common measure.

PROP. IV.



Three commensurable magnitudes being given A, B, C, *to find out their greatest common measure.*

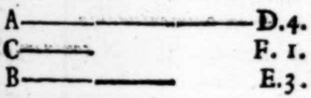
a Find out D the greatest common measure of any two A, B ; *a* also E the greatest common measure of D and C. therefore E is the magnitude sought for. a 3. 10.

a For it is clear, E measuring D and C, *b* does measure the three, A, B. C. Conceive another magnitude F greater then that to measure them ; *c* then F measures D, *c* and consequently E the greatest common measure of D, and C, the greater the lesse. *Which is Absurd.* b constr. and 2. ax. 10. c cor 3. 10.

Coroll.

Hence also it appeares , that if a magnitude measure three magnitudes, it shall likewise measure their greatest common measure.

PROP. V.



Commensurable magnitudes A , B, *have such proportion one to another , as number hath to number.*

a C being found the greatest common measure of A, B ; as often as C is conteined in A and B, so often is 1 conteined in the numbers D and E ; *b* therefore C. A :: 1. D ; wherefore inversely A. C :: D. 1. *b* but likewise C. B :: 1. E. *c* therefore of equality A. B :: D. E :: N. N. *W.W. to be Dem.* a 3. 10. b 27. def. 7.

PROP. VI.

E———— F,1.
A—————————— C,4.
B———-——— D,3.

If two magnitudes A, B, *have such proportion one to another, as number* C *hath to number* D, *those magnitudes* A, B *shall be commensurable.*

a *sch.* 10 6. What part 1 is of the number C, [a] that let E be of
b *constr.* A. Therefore because E.A [b] :: 1.C. and A.B [c] :: C.
c *hyp* D. [d] therefore of equality shall E.B :: 1. D. Where-
d 22. 5. fore seeing 1 [e] measures the number D, [f] likewise E
e 5. 4 & 7. f 20. *def.* 7. measures B; but it [g] also measures A. [h] therefore A
g *constr.* h 1. *def.* 10. ⊓⊔ B. *W.W. to be Dem.*

PROP. VII.

A——————————
B——————

Incommensurable magnitudes A, B, *have not that proportion one to another, which number hath to number.*

a 6. 10. If you affirm A.B :: N.N. [a] then A ⊓⊔ B. *against the Hyp.*

PROP. VIII.

A———-----————
B——————

If two magnitudes A, B, *have not that proportion one to another, which number hath to number, those magnitudes are incommensurable.*

a 5. 10. Conceive A ⊓⊔ B. [a] then A.B :: N.N *contrary to the Hyp.*

PROP. IX.

A————————
B————
E,4.
F,3.

The squares described of right lines commensurable in length, have that proportion one to another, that a square number hath to a square number. And squares, which have that proportion

portion one to another , that a square number hath to a square number , shall also have their sides commensurable in length. But such squares as are made of right lines incommensurable in length, have not that proportion one to another , which a square number hath to a square number. And squares which have not such proportion one to another as a square number hath to a square number, have not their sides commensurable in length.

1. *Hyp.* A. ⊐⊏ B. I say Aq.Bq :: Q. Q.

For [a] let A, B :: number E. number F. therefore $\frac{Aq}{Bq}$ [b] ($\frac{A}{B}$ twice) [c] = $\frac{E}{F}$ twice. [d] = $\frac{Eq}{Fq}$. [e] therefore Aq.Bq :: Eq·Fq :: Q.Q. *W.W. to be Dem.* [a] 5. 10. [b] 20. 6. [c] sch. 23. 5. [d] 11. 8. [e] 11. 5.

2. *Hyp.* Aq.Bq :: Eq.Fq :: Q.Q. I say A ⊐⊏ B. For $\frac{A}{B}$ twice [f] ($\frac{Aq}{Bq}$) [g] = $\frac{Eq}{Fq}$ [h] = $\frac{E}{F}$ twice. [i] therefore A. B :: E. F :: N.N. [k] wherefore A ⊐⊏ B. *Which was to be Dem.* [f] 20. 6. [g] hyp. [h] 11. 8. [i] sch. 23. 5. [k] 6. 10.

3. *Hyp.* A ⊐⊏̇ B. I deny that Aq.Bq :: Q.Q. For suppose Aq. Bq :: Q. Q. then A ⊐⊏ B, as is shewn before, *against the Hyp.*

4. *Hyp.* Not Aq. Bq :: Q. Q. I say that A ⊐⊏̇ B. For conceive A ⊐⊏ B. then Aq Bq :: Q.Q. as above, *against the Hyp.*

Coroll.

Lines ⊐⊏ are also ⊐ᒧ. but not on the contrary. And lines ⊐⊏̇ are not therefore ⊐ᒧ̇, but ⊐ᒧ̇ are also ⊐⊏̇.

PROP.

PROP. X.

If four magnitudes be proportionall (C.A :: B.D) *and the first* C *be commensurable to the second* A, *the third* B *shall be commensurable to the fourth* D. *And if the first* C *be incommensurable to the second* A, *also the third* B *shall be incommensurable to the fourth* D.

a 5. 10. C A B D If C ⊐⊏ A, a then C.A :: N.N b :: B.
b 6. 10 D. b therefore B ⊐⊏ D. But if ⊐⊏ A, c then shall not
c 7. 10. C.A :: N.N :: B.D. d wherefore B ⊐⊏ D. *W.W. to be*
d 8. 10. *Dem.*

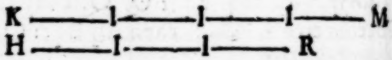
Lemma 1.

To find out two plane numbers, not having the proportion which a square number hath to a square.

Any two plane numbers not like will satisfy this Lemma, as those numbers which have superparticular, superbipartient, or double proportion; or any two prime numbers. *See schol.* 27.8.

Lemma 2.

B, 5.
C, 3.



To find out a line HR, *to which a right line given* KM *hath the proportion of two numbers given* B, C.

a *sch.* 10 6. a Divide KM into as many equall parts as there are unites in the number B. and let as many of these,
b 3 1. as there are unites in the number C, b make the right line HR. it is manifest that KM.HR :: B.C.

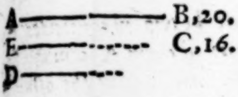
Lemma 3.

To find out a line D, *to the square of which the square of a right line given* KM *hath the proportion of two numbers given* B, C.

a 2, *lem.* 10. 10 Allow B. C a :: KM. HR. and between KM and
b 13. 6. HR, b find a mean proportionall D. Therefore
c 20 6. KMq.Dq c :: KM.HR d :: B.C.
d *constr.*

PROP.

PROP. XI.



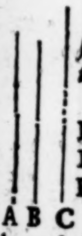
To find two right lines incommensurable to a right line given A *, one* D *in length only , the other* E *in power also.*

1. Take the numbers B, C , a so that there be not B. C :: Q. Q. b and let B. C :: Aq. Dq. c it is plane that A ⊓⊔ D. But Aq d ⊓⊔ Dq. *W.W. to be Done.* a 1.lem.10. 10. b 3.lem.10. 10.

2. d Make A.E :: E.D. I say Aq ⊓⊔ Eq. For A.D e :: Aq.Eq. therefore since A ⊓⊔ D, as before: f therefore Aq ⊓⊔ Eq. *W.W. to be Done.* c 9.10. d 6.10. d 13.6. e 10.6. f 10.10.

PROP. XII.

Magnitudes (A, B) *commensurable to the same magnitude* C, *are also commensurable one to the other.*



Because A ⊓⊔ C, and C ⊓⊔ B , a let A. C :: N. N :: D. E, & C. B :: N. N :: F. G. b take three numbers H, I, K, the least ∴ in the proportions of D to E, & F to G. Now because A.C c :: D.E c :: H. I. and C. B c :: F.G :: I. K. d therefore of equality A. B :: H. K :: N. N. e therefore A ⊓⊔ B. *W.W. to be Dem.* a 5.10. b 4.8. c constr. d 22.5. e 6. 10.

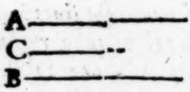
D,18. E,8,
F, 2. G,3,
H,5. I,4. K,6,

Schol.

Hence , Every right line commensurable to a rationall line is also it self rationall. And all right lines rationall are commensurable to one another, at least in power. Also, every space commensurable to a rationall space is rationall too: and all rationall spaces are commensurable one to another. But magnitudes whereof one is rationall , the other irrationall, are incommensurable amongst themselves. 12.10. and def.6. def.9. def.7. & 10.

PROP.

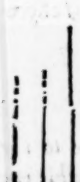
PROP. XIII.



If there be two magnitudes A, B, *and one of them* A *commensurable to a third* C, *but the other* B *incommensurable, those magnitudes* A, B *are incommensurable.*

a hyp. b 12. 10. Conceive B ⊐⊏ A. then being C [a] ⊐⊏ A, [b] therefore C ⊐⊏ B, *against the Hyp.*

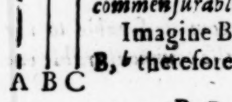
PROP. XIV.



If there be two magnitudes commensurable A, B; *and one of them* A *incommensurable to any other magnitude* C, *the other also* B *shall be incommensurable to the same* C.

a hyp. b 12. 10. Imagine B ⊐⊏ C. then for that A [a] ⊐⊏ B, [b] therefore A ⊐⊏ C, *against the Hyp.*

PROP. XV.



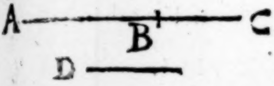
If four right lines be proportionall (A. B :: C. D) *and the first* A *be in power more then the second* B *by the square of a right line commensurable to it self in length, then also the third* C *shal be more in power then the fourth* D *by the square of a right line commensurable to it self in length. But if the first* A *be more in power then the second* B *by the square of a right line incommensurable to it self in length, then shall the third* C *be more in power then the fourth* D *by the square of a right line incommensurable to it self in length.*

a hyp. b 22. 6. c 17. 5. d 22. 6. e cor 4. 5. f 22. 5.

For because A. B [a] :: C. D. [b] therefore Aq. Bq :: Cq. Dq. [c] therefore by division Aq − Bq. Bq :: Cq − Dq. Dq. [d] wherefore √: Aq − Bq. B :: √: Cq − Dq. D. [e] and so inversely B. √: Aq − B :: D √: Cq − Dq. [f] therefore of equality A. √: Aq − Bq :: C. √: Cq − Dq. consequently if A ⊐⊏, or ⊐⊏ √ Aq − Bq,

− Bq, g then likewise C ⊓⊔, or ⊓⊔ √: Cq − Dq. g 10.10.
W.W. to be Dem.

Prop. XVI.



If two magnitudes commensurable AB, BC, *be composed, the whole magnitude* AC *shall be commensurable to each of the parts* AB, BC. *And if the whole magnitude* AC *be commensurable to either of the parts* AB, *or* BC, *those two magnitudes given at first* AB, BC, *shall be commensurable.*

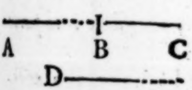
1. *Hyp.* a Let D be the common measure of AB, BC; b so D measures AC. and therefore AC ⊓⊔ AB, and BC. *Which was to be Dem.* a 3 10. b 1. ax. 10. c 1. def. 10.

2. *Hyp.* a Let D be the common measure of AC, AB. d therefore D measures AC − AB (BC) and consequently AB ⊓⊔ BC. *W.W. to be Dem.* d 3 ax. 10.

Coroll.

Hence it follows, if a whole magnitude composed of two be commensurable to any one of them, the same shall be commensurable to the other also.

Prop. XVII.



If two incommensurable magnitudes AB, BC, *be composed, the whole magnitude also* AC *shall be incommensurable to either of the two parts* AB, BC. *And if the whole magnitude* AC *be incommensurable to one of them* AB, *the magnitudes first given* AB, BC, *shall be incommensurable.*

1. *Hyp.* If it can be, let D be the common measure of AC, AB. a therefore D measures AC − AB (BC) b and therefore also AB ⊓⊔ BC, *against the Hypoth.* a 3. ax. 10. b 1 def. 10.

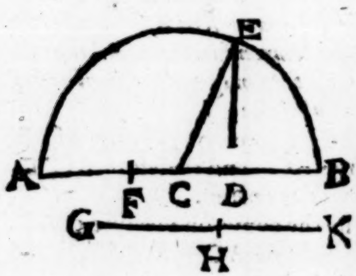
2. *Hyp.*

c 16.10. 2. *Hyp.* Conceive AB ⊓⊔ BC. *c* therefore AC ⊓⊔ AB. *Against the Hyp.*

Coroll.

Hence also, If one magnitude, composed of two, be incommensurable to any one of them, the same also shall be incommensurable to the other.

PROP. XVIII.

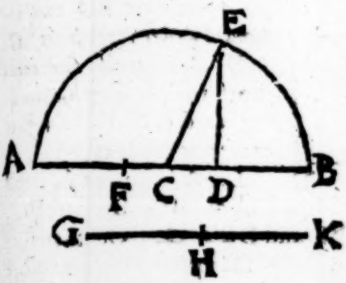


If there be two unequal right lines AB, GK, *and upon the greater* AB *a parallelogram* ADB *equall to the fourth part of a square made of the lesse line* GK, *and wanting in figure by a square, be ayplyed, and divide the said* AB *into parts commensurable in length* AD, DB; *then shall the greater line* AB *be more in power then the lesse* GK *by the square of a right line* FD *commensurable in length to the greater. And if the greater* AB *be in power more then the lesse* GK *by the square of the right line* FD *commensurable unto it self in length, and a parallelogram* ADB *equall to the fourth part of the square made of the lesse line* GK, *&* *wanting in figure by a square, be applied to the greater* AB, *then shall it divide the same into parts* AD, DB *commensurable in length.*

a 10.1. *a* Divide GK equally in H, and *b* make the rect-
b 28.6. angle ADB = GHq. Cut off AF = DB. then is ABq
c 8.2. *c* = 4 ADB *d* (4 GHq or GKq) + FDq. Now in the
d constr. and 4.2. first place, if AD ⊓⊔ DB, then shall AB *e* ⊓⊔ BD
e 16.10. *e* ⊓⊔ 2 DB *f* (AF + DB, or AB − FD) *h* therefore
g cor.16.10. AB ⊓⊔ FD. *W. W. to be Dem.* But secondly, if AB
h cor.16.10. ⊓⊔ FD, *h* then shall AB ⊓⊔ AB − FD (2 DB)
k 12.10. *k* therefore AB ⊓⊔ DB. *l* wherefore AD ⊓⊔ DB.
l 16.10. *W. W. to be Dem.*

PROP.

PROP. XIX.



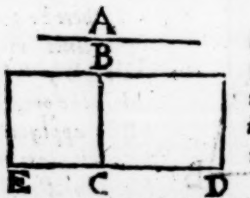
If there be two right lines unequall AB, GK, *and to the greater* AB *be applyed a parallelogram* ADB *equall to the fourth part of a square made upon the lesse* GK, *and wanting in figure by a square, and also thus applyed divide the said* AB *into parts* AD, DB *incommensurable in length; the greater line* AB *shall be in power more then the lesse* GK *by the square of the right line* FD *incommensurable to the greater in length. And if the greater line* AB *be more in power then the lesse* GK *by the square of a right line* FD *incommensurable unto it self in length, and if also upon the greater* AB *be applied a parallelogram* ADB *equall to the fourth part of the square of the lesse* GK *and wanting in figure by a square, then shall it divide the said greater line* AB *into parts incommensurable in length* AD, DB.

Suppose all the same that was done and said in the prec. prop. Therefore first, If AD ⊏⊐ DB, *a* then shall AB ⊏⊐ DB. *b* Wherefore AB ⊏⊐ 2 DB (AB − FD) *c* therefore AB ⊏⊐ FD. *W.W. to be Dem.* a 17. 10. b 13. 10. c cor 7. 10.

Secondly, If AB ⊏⊐ FD, then AB ⊏⊐ AB − FD (2 DB;) *d* wherefore AB ⊏⊐ DB, *e* and consequently AD ⊏⊐ DB. *W.W. to be Dem.* d 13. 10. e 17. 10.

PROP.

PROP. XX.



A rectangle BD *comprehended under right lines* BC, CD, *rationall and commensurable in length, according to one of the foresaid wayes, is rationall.*

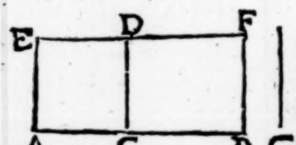
Let A be given ρ, and

a 46. 1. ^a^ the square BE described upon BC. Because DC.
b 1. 6. CE (BC) ^b^ :: BD. BE. and DC ^c^ ⊐⊏ BC, ^d^ therefore
c hyp. shall the rectangle BD be ⊐⊏ square BE. wherefore
d 10. 10. seeing the square BE ^e^ ⊐⊏ Aq, shall also BD be
e hyp. and 9. def. 10. ⊐⊏ Aq. and so the rectangle BD is ^f^ ρ. *W. W. to be Dem.*
f 12. 12.

Note. *There are three kinds of lines rationall commensurable one to another. For either, of two lines rationall commensurable in length one to the other, one is equall to the rationall line propounded, or neither of them is equall to it, notwithstanding both of them are commensurable to it in length; or lastly both of them are commensurable to the rationall line given only in power. And these are the wayes which the present theoreme speaks of.*

In numbers, let there be BC, $\sqrt{8}$ ($2\sqrt{2}$) and CD $\sqrt{18}$ ($3\sqrt{2}$) then shall the rectangle BD $= \sqrt{144} = 12$.

PROP. XXI.



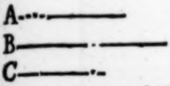
If a rationall rectangle DB *be applyed to a rationall line* DC, *it makes the breadth thereof* CB *rationall, and commensurable in length to that line* DC, *whereto* DB *is applyed.*

a 1. 6. Let G be propounded ρ, and the square DA de-
b hyp. scribed on BC. because BD. DA :: ^a^ BC. CA; and
c sch 12. 10. BD, DA ^b^ are ρa ^c^ & so ⊐⊏. ^d^ therefore BC ⊐⊏ CA.
d 10. 10.
e sch. 12. 10. but CD (CA) is ρ. ^e^ therefore BC is ρ. *W. W. to be Dem.*

In

In numbers, let there be the rectangle DB, 12. and DC, √8. then shall CB, √18. but √18 = 3 √2. and √8 = 2 x √2.

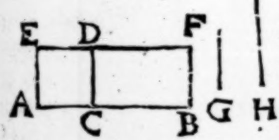
Lemma.



To find out two right lines rationall commensurable only in power.

Let A be propounded ρ'. a Take B ⊐⊏ A, a and C ⊐⊏ B. b it is clear that B and C are the lines required. a 11. 10. b sch. 12. 10.

PROP. XXII.



A rectangle DB *comprehended under right lines rationall* DC, CB *commensurable in power only, is irrationall: & the right line* H, *which conteineth that rectangle in power is irrationall, and called a Mediall line.*

Let G be the propounded ρ', and the square DA described on DC, and let Hq = DB. Because AC. CB a :: DA. DB. b & AC ⊐⊏ CB, c shall be DA ⊐⊏ DB (Hq.) d but Gq ⊐⊏ DA. e therefore Hq ⊐⊏ Gq. f wherefore H is ρ'. *Which was to be Dem.* and let it be called a Mediall line, because AC. H :: H. CB.

a 1. 6. b hyp. c 10. 10. d hyp. and 9. 3ef. 10. e 13. 10. f 11. 10.

In numbers, let there be DC, 3. and CB, √6. then shall the rectangle be DB (Hq) √54. wherefore H is √√54.

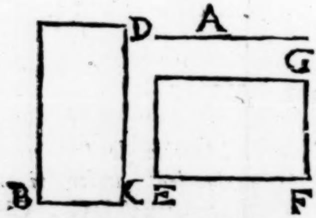
The note of a mediall line is μ, of a mediall rectangle μν, of more together μα.

Schol.

Every rectangle that can be conteined under two right lines rationall commensurable only in power, is mediall, although it be conteined under two right lines irrationall: and every mediall rectangle may be conteined under two right lines rationall, commensurable only in power. as for example, the

$\surd$ 24 is $\mu\nu$, because it is conteined under $\surd$ 3, and $\surd$ 8, which are $\acute{\rho}$, ⊤. although it may be conteined under $v\surd$ 6, and $v\surd$ 96 irrationalls; for $\surd 24 = v\surd 576 = v\surd 6 \times v\surd 96$.

PROP. XXIII.

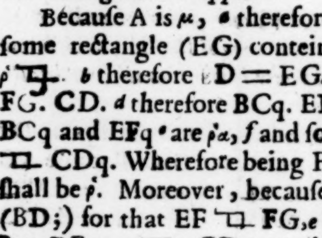


If the rectangle BD *made of a mediall line* A, *be applied on a rationall line* BC, *it makes the breadth* CD *rationall, and incommensurable in length to the line* BC, *whereunto the rectangle* BD *is applied.*

a sch.12.10.
b 1.ax.1.
c 14.6.
d 22.6.
e hyp.
f sch.12.10.
g 10.10.
h sch.12.10.
k 1.6.
l 10.10.
m sch.12.10.
n 13.10.
o 1.6.
p 10.10.

Because A is μ, *a* therefore shall Aq be equall to some rectangle (EG) conteined under EF and FG $\acute{\rho}$ ⊤. *b* therefore BD = EG. *c* whence BC. EF :: FG. CD. *d* therefore BCq. EFq :: FGq. CDq. But BCq and EFq *e* are $\acute{\rho}\alpha$, *f* and so ⊓. *g* therefore FGq ⊓ CDq. Wherefore being FG is $\acute{\rho}$, *h* therefore CD shall be $\acute{\rho}$. Moreover, because EF.FG *k* :: EFq.EG (BD;) for that EF ⊓ FG, *e* shall EFq be ⊓ BD. But EFq *m* ⊓ CDq. *n* therefore the rectangle BD ⊓ CDq. Whence being CDq. BD *o* :: CD.BC. *p* shall CD be ⊓ BC. therefore, &c.

PROP. XXIV.



A right line B *commensurable to a mediall line* A *is also a mediall line.*

a 11. 6.
b hyp.
c 23.10.

Upon CD $\acute{\rho}$ *a* make the rectangle CE = Aq; *a* and the rectangle CF = Bq. Because Aq (CE) is $\mu\nu$, *b* and CD $\acute{\rho}$, *c* therefore shall the latitude DE be $\acute{\rho}$ ⊤ CD. But for

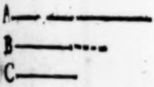
for that CE. CF [d] :: ED. DF. and CE [e] ⊤⊥ CF, [f] therefore ED ⊤⊥ DF. [g] therefore DF is ρ ⊤⊥ CD. [h] whence the rectangle CF (Bq) is $\mu\nu$, and so B is μ. *W.W. to be Dem.* d 1. 6. e hyp. f 10. 10. g 12. and 13. 10. h 22. 10.

Obs. *that the note* ⊤J *for the most part signifies commensurable in power only, as in this and the precedent demonstrations*, &c.

Coroll.

Hereby it is manifest that a space commensurable to a mediall space, is also mediall.

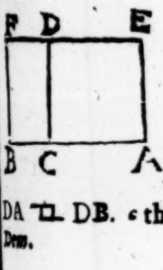
Lemma.



To find out two right lines mediall A, B, *commensurable in length, and also two,* A, C, *commensurable only in power,*

[a] Let A be any μ, [b] take B ⊤⊥ A, and [c] C ⊤J A. [d] it appears to be done. a lem. 22. 10. and 13 6. b 2. lem. 10. 10. c 3. lem. 10. 10. d constr. and 24. 10.

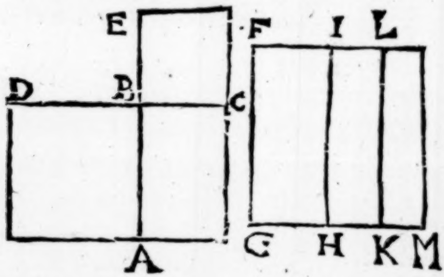
PROP. XXV.



A rectangle DB *conteined under* DC, CB *mediall right lines commensurable in length, is mediall.*

Upon DC describe the square DA. Being AC (DC.) CB [a] :: DA. DB, & DC ⊤⊥ CB; [b] shall DA ⊤⊥ DB. [c] therefore DB is $\mu\nu$. *Which was to be Dem.* a 1. 6. b 10. 10. c 24. 10.

Prop. XXVI.

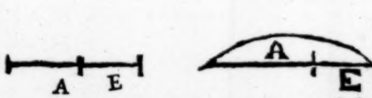


A rectangle AC *comprehended under mediall right lines* AB, BC *commensurable only in power, is either rationall or mediall.*

a 46. 1. b cor. 16. 6. — Upon the lines **AB**, BC, *a* describe the squares **AD**,**CE**; and upon FG ρ' *b* make the rectangles FH, = AD, *b* and IK = AC, *b* and LM = CE.

c hyp. & 14. 10. d 23. 10. e 10. 10. f 20. 10. g sch. 22. 6. h 1. 6. k 17. 6. l 12. 10 m 20. 10. n 22. 10. — The squares AD, CE, that is, the rectangles FH, LM, *c* are μα and ⊏⊐. therefore GH, KM, having the same proportion *d* are ρ', *e* and ⊏⊐. *f* therefore GH x KM is ρ'ν. But because AD, AC, CE, that is, FH, IK, LM, *g* are ∺; *h* and so GH, HK, KM also ∺; *k* thence HKq = GH x KM. *l* therefore HK is ρ', or ⊏⊐, or ⊏⊐̲ IH (GF;) if ⊏⊐, *m* then the rectangle IK or AC is ρ'ν. but if ⊏⊐̲, *n* then AC is μν. *Which was to be Dem.*

Lemma.



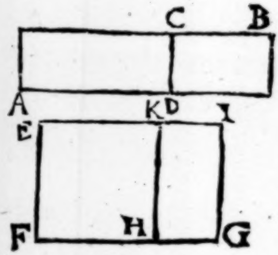
* *If* A *and* E *be* ⊏⊐̲ *only, Then first,* shall Aq, Eq, Aq + Eq, Aq − Eq *a* ⊏⊐. *And secondly,* Aq, Eq, Aq + Eq, **Aq** − Eq ⊏⊐ AE and 2 E.

a hyp. and 16. 10. b 1. 6. c hyp. d 10. 10. e 14. 10. f 14. 10. — For **A**.E *b* :: Aq. AE *b* :: AE. Eq. therefore seeing A *c* ⊏⊐ E, *d* shall Aq ⊏⊐ AE, *e* and 2 AE. also Eq *d* ⊏⊐ AE, *e* and 2 AE. wherefore because Aq + Eq ⊏⊐ Aq and Eq; and Aq − Eq ⊏⊐ Aq and Eq. *f* therefore shall **Aq** + Eq, *f* and Aq − Eq be ⊏⊐ AE, and 2 AE.

Hence

Hence also thirdly, Aq, Eq; Aq + Eq, Aq − Eq, 2 AE *g* ˙⊐⊏ Aq + Eq + 2 AE; and Aq + Eq − 2 AE. *g* and Aq + Eq + 2 AE ˙⊐⊏ Aq + Eq − 2 AE. *h* (Q. A − E.) *g* 14.16 *an.* 17.10. *h cor.* 7. 12.

Prop. XXVII.



A mediall rectangle AB *exceedeth not a mediall rectangle* AC *by a rationall rectangle* DB.

Upon EF ρ', *a* make EG = AB, *a* and EH = AC. The rectangles AB, AC, *i.e.* EG, EH, *b* are μα; *c* therefore FG and FH are ρ' ⊐⊏ EF. *a cor.* 16.6. *b hyp.* *c* 23.10.

Whence, if KG, *d i.e.* DB be ρ'ν, *e* then shall HG be ⊐⊏ HK; *f* wherefore HG ⊐⊏ FH. *g* and consequently FGq ˙⊐⊏ FHq. But FH is ρ'. *h* therefore is FG ρ. but FG was ρ' before. *Which is contradictory.* *d* 3. *ax* 1. *e* 21. 10. *f* 13. 10. *g lem.* 16. 10. *h sch.* 12. 10.

Schol.

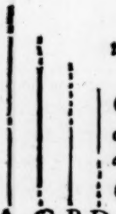
1. *A rationall rectangle* AE *exceeds a rationall rectangle* AD *by a rationall rectangle* CE.

For AE *a* ⊐⊏ AD, *b* therefore AE ⊐⊏ CE. *c* wherefore CE is ρ'ν. Which, &c. *a hyp.* *b cor.* 16. 10. *c sch.* 12. 10.

2. *A rationall rectangle* AD *joined with a rationall rectangle* CF *makes a rationall rectangle* AF.

For AD *a* ⊐⊏ CF. *b* wherefore AF ⊐⊏ AD and CF; *c* and so AF is ρ'ν. *W. W. to be Dem.* *a sch.* 12. 10. *b* 16. 11. *c sch.* 12. 10.

PROP. XXVIII.



To find out mediall lines (C and D,) which contain a rationall rectangle CD.

a lem.21 10. b 13.6. c 12.6. d 22.10. e constr. f 10.10. g 24.10. h 17.6. k sch.12.10.

[a] Take A and B ρ ⊤⊥. [b] make A. C :: C.B. [c] and A.B :: C.D. I say the thing required is done. For AB (Cq) [d] is μρ, [d] whence C is μ. but being that A. B [e] :: C.D. [f] therefore C ⊤⊥ D. [g] and consequently D is μ. Moreover by inversion A.C :: B.D.*i.e.* C. B :: B.D. [h] therefore Bq = CD. But Bq is ρ. [k] therefore C D is ρ. *W.W.to be done.*

In numbers, let A be $\surd$ 2; and B $\surd$ 6. therefore C is $u\surd$ 12. make $\surd$ 2. $\surd$ 6 :: $u\surd$ 12. D. or $u\surd$ 4. $u\surd$ 36 :: $u\surd$ 12. D. then shall D be $u\surd$ 108. but $u\surd$ 12 x $u\surd$ 108 = $u\surd$ 1296 = $\surd$ 36 = 6. therefore CD is 6. likewise C.D :: 1. $\surd$ 3. wherefore C ⊤⊥ D.

PROP. XXIX.

To find out mediall right lines commensurable in power only, D *and* E, *containing a mediall rectangle* DE.

a lem 21.10. b 13 6. c 12. 6. d 17 6. e 22.10. f constr. g 10.10. h 24. 10. k constr. and cor 4 5. l 16 6. m 22.6.

[a] Take A, B, C, ρ ⊤⊥. make A. D [b] :: D.B. [c] and B. C :: D.E. I say the thing desired is performed.

For AB [d] = Dq. and AB [e] is μρ. therefore D is μ; and B [f] ⊤⊥ C, [g] whence D ⊤⊥ E. therefore [h] E is μ. Moreover B.C [f] :: D.E. and by inversion B.D :: C. E.i.e. D.A :: C.E. [l] therefore DE = AC. But AC [m] is μρ. therefore DE is μρ. *W.W.to be Done.*

In numbers, let A be 20. and B, $\surd$ 200, and C, $\surd$ 80. Therefore D is $\surd\surd$ 80000; and E $u\surd$ 12800. Therefore DE = $\surd\surd$ 1024000000 = 32000. and D.E :: $\surd$ 10. 2. wherefore D ⊤⊥ E.

Schol.

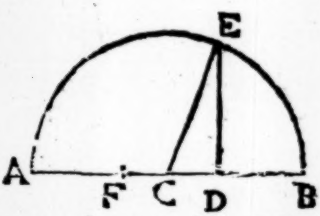
Schol.

To find out two plane numbers, like or unlike.

Take any four numbers proportionall A.B :: C.D. it is manifest that AB and CD are like plane numbers. And you may find out as many unlike plane numbers, as you please, by help of *Schol.* 27.8.

A,6.	C,12.
B,4.	D,8.
$\overline{AB}$, 24.	$\overline{CD}$,96.
A,6.	C,5.
B,4.	D,8.
$\overline{AB}$, 24.	$\overline{CD}$,40.

Lemma.



To find out two square numbers (DEq *and* CDq) *so that the number composed of them* (CEq) *be square also.*

Take AD, DB like plane numbers (of which let both be equall, or both odde) *viz.* AD,24. and DB, 6. The totall of these (AB) is 30; the difference (FD) 18. half of which (CD) is 9. *a* Now the like plane numbers AD, DB, have one mean number proportionall, namely DE. therefore it is evident that every of those numbers CE, CD, DE, are rationall, and by consequence CEq (*b* CDq + DEq) is the square number required. a 18. 8. b 47. 1.

Whereby it will be easy to find out two square numbers, the excesse of which is a square or not a square number. namely by the same construction *c* shall CEq − CDq be = DEq. c 3. ax. 1.

But if AD, DB be plane numbers unlike, the mediall

diall proportionall line (DE) shall not be a rationall number, and so neither shall the excesse (DEq) of the square numbers, CEq, CDq, be a square number.

Lemma 2.

2. *To find out two such square numbers* B, C, *as the number compounded of them* D *is not square. Also to divide a square number* A *into two numbers* B, C, *not squares.*

A, 3. B, 9. C, 36. D, 45.

1. Take any square number B, and let C be = 4 B, and D = B + C. I say the matter is done.

For B is Q. by the constr. likewise because B.C :: 1.4 :: Q.Q. *a* therefore C also shall be a square number. But because B + C. (D) C :: 5. 4 :: not Q. Q. *b* therefore shall not D be a square number. *W.W. to be done.*

a 24. 8.
b cor, 24. 8.

A, 36. B, 24. C, 12. D, 3. E, 2. F, 1.

2. Let A be some square number. Take D, E, F, plane numbers unlike, and let D be = E + F. make D.E :: A. B. and D. F :: A. C. I say the thing required is done.

For because D.E + F :: A.B + C. and D = E + F, *a* therefore shall A = B + C. Now suppose B to be square, *b* then A and B, *c* and consequently D and E are like plane numbers. *Which is contrary to the Hyp.*

a 14. 5.
b 21. def. 7.
c 26. 8.

The same absurdity will follow if C be supposed a square number. Therefore, &c.

PROP.

PROP. XXX.

C .... E ..... D

To find out two such rationall right lines AB, AF, *commensurable only in power, as the greater* AB *shal be in power more then the lesse* AF *by the square of a right line* BF *commensurable in length to the greater.*

Let AB be the line given ρ'. *a* Take the square numbers CD, CE, so that CD − CE (ED) be not Q *b* and let there be CD. ED :: ABq. AFq. In a circle described upon the diameter AB *c* draw AF, and also BF. Then I say AB, AF, are the lines required. a 1.lem.29. 10. b 3.lem.19. 10. c 1. 4.

For ABq. AFq *d* :: CD. ED. *e* therefore ABq ⊓⊔ AFq. but AB is ρ'. *f* therefore AF is also ρ'. But because CD is Q: and ED not Q: *g* therefore shall AB be ⊓⊔ AF. Moreover by reason of the *h* right angle AFB, is ABq *k* = AFq + BFq; therefore seeing ABq. AFq :: CD. ED. by conversion of proportion shall ABq. BFq :: CD. CE :: Q. Q. *l* therefore AB ⊓⊔ BF. *W. W. to be Done.* d constr. e 6. 10. f sch.12. 10. g 9. 10. h 31. 3. k 47. 1. l 9. 10.

In numbers, let there be AB, 6; CD, 9; CE, 4; wherefore ED, 5. Make 9.5 :: 36. (Q:6) AFq. then AFq shall be 20. and consequently AF √ 20. therefore BFq = 36 − 20 = 16. wherefore BF is 4.

PROP. XXXI.

C .... E .... D

To find out two rationall lines AB, AF *commensurable only in power, so that the greater* AB *shall be in power more then the lesse* AF *by the square of a right line* BF *incommensurable in length to the greater.*

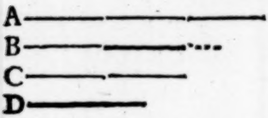
Let AB be the line given ρ'. *a* Take the square numbers CE, ED, so that CD = CE + ED be not Q. and in the rest follow the construction of the preced. *prop.* I say then the thing required is done. a 1.lem.29. 10.

For

For, as above, AB, AF, are ρ ⊤ᒧ. also ABq. BFq ::
b 9. 10. CD. ED. therefore being CD is not Q. AB, BF[b] shall be ⊤⊔. *Which was to be Done.*

In numbers, let there be AB, 5. CD, 45. CE = 36. ED = 9. Make 45. 9 :: 25 (ABq.) 5 (AFq.) therefore AF = √ 5. consequently BFq = 45 − 25 = 20. wherefore BF = √ 20.

PROP. XXXII.

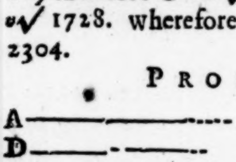


To find out two mediall lines C, D, *commensurable only in power, comprehending a rational rectangle* CD, *so that the greater* C *be more in power then the lesser* D *by the square of a right line commensurable in length to the greater.*

a 30. 10. [a] Take A and B ρ ⊤ᒧ; so as √ Aq − Bq ⊤⊔ A.
b 13. 6. [b] and make A. C :: C. B.[c] and A. B :: C. D. I say the
c 12. 6. thing is done.
d constr. For because A and [d] B are ρ ⊤ᒧ. [e] therefore shall
e 22. 10. C ([f] √ AB) be μ. [g] & thence also C ⊤ᒧ D.[h] there-
f 17. 6. fore D is likewise μ. Furthermore, whereas A. B [d] ::
g 10. 10. C. D; and inversely A. C :: B. D :: C. B; and Bq is ρ.
h 24. 10. therefore shall CD ([k] Bq) be ρ. Lastly, because √
k 17. 6. Aq − Bq [d] ⊤⊔ A, [l] shall √ Cq − Dq be ⊤⊔ C. there-
l 15. 10. fore, &c. But if √ Aq − Bq ⊤⊔ Aq, then shall √ Cq − Dq be ⊤⊔ C.

In numbers; let there be A 8, B √ 48 (√: 64 − 16) therefore C = √ AB = υ√ 3072. and D = υ√ 1728. wherefore CD = υ√ 5308416 = √ 2304.

PROP. XXXIII.



To find out two mediall lines D, E, *commensurable in power only, comprehending a mediall rectangle* DE, *so that the greater* D *shall be more in power then the lesse* E, *by the square of a right line commensurable to the greater in length.*

[a] Take

a Take A and C ρ' ⊓, so that $\sqrt{}$ Aq − Cq ⊏ A. *b* take also B ⊓ A and C, and make A.D *c* :: D. B *d* :: C.E. then D, and E are the lines sought for.

For because A and C *e* are ρ', *e* and B ⊓ A and C, *f* therefore shall B be ρ', and D ($\sqrt{}$ AB) *g* shall be μ. But because A.D :: C.E, therefore inversely A.C :: D.E. wherefore seeing A ⊓ C, therefore D shall be ⊓ E. therefore E is μ. Furthermore, *l* being D.B :: C.E. and BC is $\mu\nu$. also DE, equall to it, is $\mu\nu$. Lastly because A. C :: D. E. *e* seeing $\sqrt{}$ Aq − Cq ⊏ A. therefore $\sqrt{}$ Dq − Eq ⊏ D. therefore, &c. But if $\sqrt{}$ Aq − Cq ⊏ A. then $\sqrt{}$ Dq − Eq ⊏ Eq.

In numbers, let there be A 8, C $\sqrt{}$ 48. B $\sqrt{}$ 28. then D $v\sqrt{}$ 3072. and E $v\sqrt{}$ 588. wherefore D.E :: 2.$\sqrt{}$ 3. and DE = $\sqrt{}$ 1344.

a 30.10. b lem. 21.10. c 13.6. d 12.6. e constr. f sch. 12.10. g 22.10. h 10.10. k 24.10. l 22.10. m 16.6. n 15.5.

PROP. XXXIV.

To find out two right lines AF, BF, *incommensurable in power, whose squares added together make a rationall figure, and the rectangle contained under them mediall.*

a Let there be found AB, CD, ρ' ⊓; so that $\sqrt{}$ ABq − CDq ⊏ AB. divide CD equally in G. *b* make the rectangle AEB = GCq. Upon AB the diameter draw the semicircle AFB, erect the perpendicular EF, and draw AF, BF. These are the lines required.

For AE.BE *d* :: BA x AE. AB x BE. But BA x AE *e* = AFq; and AB x BE = FBq. *f* therefore AE.EB :: AFq. FBq. therefore being AE *g* ⊏ EB, *h* AFq shall be ⊏ FBq. Moreover ABq (*k* AFq + FBq) *l* is ρ'. Lastly EFq *l* = AEB *l* = CGq. *m* therefore EF = CG. therefore CD x AB = 2 EF x AB. But CD x AB *n* is $\mu\nu$. *o* therefore AB x EF, *p* or AF x FB, is $\mu\nu$. *W.W. to be Dem.*

a 31.10. b 10.1. c 28.6. d 12.6 e cor. 8.6. & 17.6. f 7.5. g 19.10. h 10.10. k 31.3. and 47.1. l constr. m 1. ax. 1. n 22.10. o 24.10. p sch. 22.6.

The

The Explication of the ſame by numbers.

Let AB be 6. CD $\sqrt{}$ 12. then CG $= \sqrt{\frac{14}{4}} = \sqrt{3}$. But AE $= 3 + \sqrt{6}$. and EB $= 3 - \sqrt{6}$. whence AF ſhall be $\sqrt{:18 + 216}$. and FB $\sqrt{:18 - \sqrt{216}}$. Alſo AFq + FBq is 36, and AF x FB $= \sqrt{108}$.

But AE is found in this manner. Becauſe BA (6.) AF :: AF AE. therefore 6 AE = AFq = AEq + 3 (EFq.) therefore 6 AE − AEq = 3. Put 3 + e = AE. then 18 + 6e − 9 − 6e − ee, that is 9 − ee = 3. or ee = 6. wherefore e $= \sqrt{6}$. & ſo AE $= 3 + \sqrt{6}$.

PROP. XXXV.

To find out two right lines AE, EB, *incommenſurable in power, whoſe ſquares added together make a mediall figure, and the rectangle contained under them rationall.*

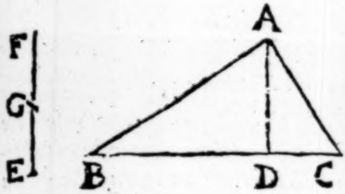
a 32.10. *a* Take AB and CF μ ⊤, ſo that AB x CF be ῥν, and $\sqrt{}$ ABq − CFq ⊓⊔ AB. and let the reſt be done as in the prec. *prop.* AE, EB are the lines required.

For, as it is ſhewn there, AEq ⊓⊔ EBq. alſo ABq (AEq + EBq) is *μν.* and laſtly AB x CF *b* is ῥν. *c* therefore alſo AB x DE, that is, AE x EB, is ῥν. therefore, &c.

b *conſtr.*
c *ſch.*12.10.
d *ſch.*22.6.

PROP.

PROP. XXXVI.

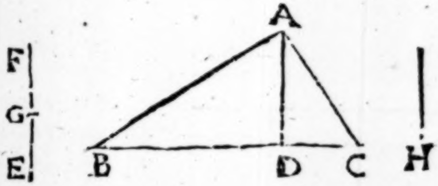


To find out two right lines BA, AC, *incommensurable in power, whose squares added together make a mediall figure, & the rectangle also contained under them mediall, and incommensurable to the figure composed of the squares.*

a Take BC and EF μ ⊤⊥, so that BC x EF be μν. and √ BCq − EFq ⊓⊔ BC. and so forward, as in the prec. BA, AC, shall be the lines sought for. a 33, 10

For (as above) BAq ⊓⊔ ACq. also BAq + ACq is μν. and BA x AC is μν. Lastly BC *b* ⊓⊔ EF, and *c* so BC ⊓⊔ EG; likewise BC. EG *d* :: BCq. BC x EG (BC x AD, or BA x AC) *e* therefore BCq (ABq + ACq) ⊓⊔ BA x AC. therefore, &c. b *constr.* c 13. 10. d 1. 6. e 14 10.

Schol.



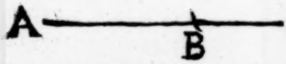
To find out two mediall lines incommensurable both in length and power.

a Take BC μ. and let BA x AC be μν, and ⊓⊔ BCq (BAq + ACq) *b* make BA. H :: H. AC. then I say BC & H are μ ⊤⊥. For BC is μ. *a* and BA x AC (*c* Hq) is μν. wherefore H is also μ. *d* Likewise BA x AC ⊓⊔ BCq; therefore Hq ⊓⊔ BCq. therefore, &c. a 36. 10. b 33. 6. c 17. 6. d 14. 10.

Here

Here begin the senaries of lines irrationall by composition.

PROP. XXXVII.



If two rationall lines AB, BC, *commensurable only in power, be added together, the whole line* AC *is irrationall, and is called a binomiall line, or of two names.*

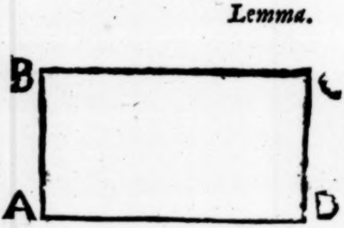
a hyp. For because AB ^{a} └□┘ BC, thence ^{b} shall ACq
b lem.26.10. be └□┘ ABq. But AB ^{a} is ρ′. ^{c} therefore AC is ′ρ. *Which*
c 11.def.10. *was to be Dem.*

PROP. XXXVIII.

If two mediall lines AB, BC, *only in power commensurable, be compounded, and contain a rationall rectangle, the whole line* AC *is irrationall, and called a first bimediall line.*

a hyp. For being that AB ^{a} └□┘ BC, ^{b} shall ACq be
b lem.26.10. └□┘ AB x BC, ρ′v. ^{c} therefore AC is ′ρ. *W.W. to be Dem.*
c 11.def.10.

Lemma.

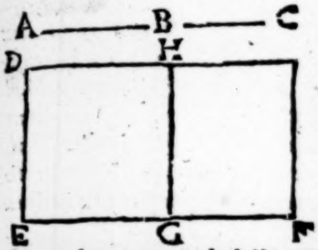


A rectangle AC, *conteined under a rationall line* AB *and an irrationall line* BC, *is irrationall.*

For if the rectangle AC be ^{a} af-
a hyp. firmed ρ′v, ^{a} then being AB is ρ′, ^{b} the breadth BC
b 21.10. shall be also ρ′. *against the Hyp.*

PROP.

PROP. XXXIX.



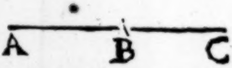
If two mediall lines AB, BC, *commensurable only in power, containing a mediall rectangle, be compounded, the whole line* AC *shal be irrationall, and is called a Second bimediall line.*

Upon the propounded line DE ρ' [a] make the rectangle DF = ACq; [b] and DG = ABq + BCq.

Because ABq [c] ˥⊔ BCq, [d] therefore ABq+BCq, *i.e.* DG, ˥⊔ ABq: but ABq [e] is μν, *e* therefore DG is μν. But the rectangle ABC is taken μν; *e* and consequently 2 ABC ([f] HF) is μν. [g] therefore EG and GF are ρ'. Being also that DG [h] ˥⊔ HF; and DG. HF :: [k] EG. GF; [l] therefore EG ˥⊔ GF. [m] therefore the whole EF is ρ'. [n] wherefore the rectangle DF is ρ'ν. [o] therefore $\surd$ DF, *i.e.* AC, is ρ'. *W. W. to be Dem.*

a *cor* 16.5.
b 47.1, *and* 11.6.
c *hyp.*
d 16.10.
e 24.10.
f 4.2.
g 27.10.
h *lem.* 26.10.
k 1.6.
l 10.10.
m 37.10.
n *lem.* 33.10.
o 11. *def.* 10.

PROP. XXXX.



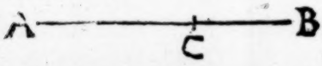
If two right lines AB, BC *commensurable only in power, be added together, making that which is composed of their squares rationall, and the rectangle conteined under them mediall, the whole right line* AC *is irrationall and is called a Major line.*

For whereas ABq + BCq [a] is ρ'ν. and [b] ˥⊔ 2 ABC [c] μν; and so ACq ([d] ABq + BCq + 2 ABC) [e] ˥⊔ ABq + BCq ρ'ν [f] therefore shall AC be ρ'. *Which was to be Dem.*

a *hyp.*
b *sch.* 12.10.
c *hyp.* & 24.10.
d 4.2.
e 17.10.
f 11. *def.* 10.

PROP.

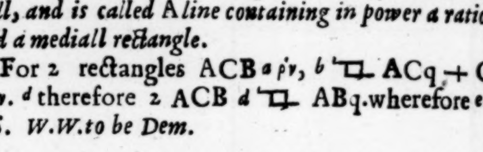
PROP. XLI.



If two right lines AC,CB,*incommensurable in power, be added together, having that which is made of their squares added together mediall, and the rectangle conteined under them rationall, the whole right line* AB *shall be irrationall, and is called* A *line containing in power a rationall and a mediall rectangle.*

a hyp. and sch 12.10. For 2 rectangles ACB a ρ́ν, b ⊐⊏ ACq + CBq
b sch.12.10 c μν. d therefore 2 ACB d ⊐⊏ ABq. wherefore e AB
c hyp. is ρ́. *W.W. to be Dem.*
d 17.10.
e 11. def. 10.

PROP. XLII.



If two right lines GH, HK, *incommensurable in power be added together, having both that which is composed of their squares mediall, and the rectangle contained under them mediall, and incommensurable to that which is composed of their squares, the whole right line* GK *is irrationall, & is called* A *line containing in power two mediall figures.*

Upon the propounded line FB ρ́ make the rectangles AF = GKq, & CF = GHq + HKq. Being GHq
a hyp. + HKq (CF) a is μν, the breadth CB b shall be ρ́.
b 23.10. Also because 2 rectangles GHK (c AD) a is μν, there-
c 4.2. fore AC b shall be ρ́. Moreover because the rectangle
d 1.6. AD a ⊐⊏ CF, d and AD. CF :: AC. CB. e thence
e 20.10. shall e AC be ⊐⊏ CB. f wherefore A is g ρ́. therefore
f 37.10. the rectangle AF. *i.e.* GKq is ρ́ν; h and consequent-
g lem.38.10. ly GK is ρ́. *W.W. to be Dem.*
h 21 def.10.

PROP.

PROP. XLIII.

A line of two names, or binomiall, AB *, can at one point only* D *be divided into its names* AD, DB.

If it be possible, let the binomiall line AB be divided at the point E, into other names AE, EB. It is manifest that the line AB is in both cases divided unequally, since AD ⊐ DB, and AE ⊐ EB.

Because the rectangles ADB, AEB *a* are μας *a* and each of ADq, DBq, AEq, EBq is ρ'α. *b* and so ADq +DBq *b* and AEq + EBq are also ρ'α. *b* therefore ADq + DBq −: AEq + EBq. *i. e.* 2 AEB − 2 ADB is ρ'ν. *d* therefore AEB − ADB is ρ'ν. therefore μν exceeds μν by ρ'ν. *e Which is Absurd.* a 37. 10. b sch. 27. 10. c sch 5. 2. d sch. 12. 10. e 27. 10.

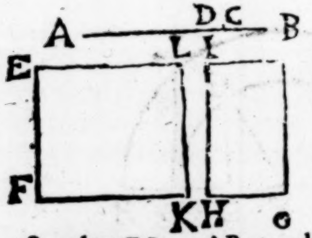
PROP. XLIV.

A first bimediall line AB*, is in one point only* D *divided into its names* AD, DB.

Conceive AB to be divided into other names AE, EB. whereupon every one ADq, DBq, EBq, will be *a* μα. and the rectangles ADB, AEB, and the doubles of them, ρα. *b* therefore 2 AEB − 2 ADB. *c i, e.* ADq + DBq −: AEq + EBq is ρ'ν. *W. is Abs.* a 38. 10. b sch. 27. 10. c sch 5. 2. d 27. 10.

P PROP.

PROP. XLV.



A second bimediall line AB, *is divided into its names* AC, CB, *onely at one point* C.

Suppose there were other names AD, DB. Upon the propounded line EF ρ′ make the rectangles EG = ABq, and EH = ACq + CBq. as also EK = ADq + DBq.

a 39. 10.
b 16 *and* 24. 10
c 23 10.
d 24. 10.
e 4. 2.
f lem. 26. 10.
g 1. 6.
h 10. 10.
k 37. 10.

Because ACq, BCq *a* are μα ⊏⊐, *b* ACq + CBq (EH) shall be μν. *c* therefore the breadth FH is ρ′. *a* moreover the rectangle ACB, *d* and so 2 ACB (*e* IG) is μν. *c* therefore HG is also ρ′. And since EH is *f* ⊏⊐ IG, *g* & EH.IG :: FH.HG. *h* therefore FH, HG shall be ⊏⊐. *k* therefore FG is a binomial, whose names are FH, HG. By the same reason FG is binomiall, and the names of it FK, KG; *contrary to the* 43. *of this Book*.

PROP. XLVI.

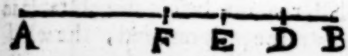
A Major line AB *is at one point only* D *divided into its names* AD, DB.

a 40 10.
b *sch* 27 10.
c *sch* 5. 2.
d 27 10.

Imagine other names AE, EB. whereupon the rectangles ADB, AEB, are *a* μα. *a* and as well ADq + DBq, as AEq + EBq are ρ′α. *b* therefore ADq + DBq − : AEq + EBq, *c* i.e. 2 AEB − 2 ADB is ρ′. *d Which is impossible*.

PROP.

PROP. XLVII.

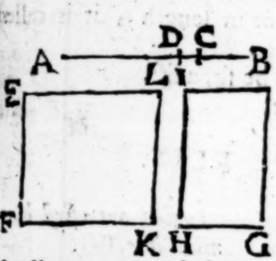


A line AB *containing in power a rationall and a mediall figure is divided at one point only* D *into its names* AD, DB.

Conceive other names AE, EB. then both AEq + EBq, and ADq + DBq are μα. a and the rectangles AEB, ADB are ρ'α. b therefore 2 AEB − 2 ADB, c i. e. ADq + DBq −: AEq + EBq is ρ'γ. d *Which is Absurd.*

a 41 10.
b sch. 27. 10.
c sch. 5. 2.
d 27. 10.

PROP. XLVIII.



A line AB *containing in power two mediall rectangles, is at one point only* C *divided into its names* AC, CB.

If you would divide AB into other names AD, DB. draw upon the line propounded EF ρ' the rectangles EG = ABq, and EH = ACq + CBq, and EK = ADq + DBq. then because ACq + CBq, namely EH, a is μν, b the breadth FH shall be ρ'. Also because 2 ACB, c that is, IG, is d μν, HG b shall be likewise ρ'. Therefore, whereas EH ⊐⊏ IG. and EH. IG d :: FH. HG, thence FH e shall be ⊐⊏ HG. f therefore FG is a binomiall, and the names of it FH, HG. In like manner FK, KG shall be the names of it, *against the 43. of this Book.*

a 42. 10.
b 23. 10.
c 4. 2.
d 1. 6
e 10. 10.
f 37. 10.

Second Definitions.

A Rationall line being propounded, and the binomiall, divided into its names, the greatest of whose names is more in power then the lesse by a square of a right line commensurable to the greater in length; then

 1. If

I. If the greater name be commensurable in length to the rationall line propounded, the whole line is called a first binomiall line.

II. But if the lesser name be commensurable in length to the rationall line propounded, the whole line is called a second binomiall.

III. If neither of the names be commensurable in length to the rationall line propounded, it is called a third binomiall.

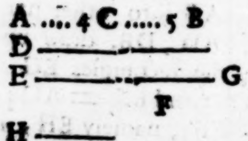
Furthermore, if the greater name be more in power then the lesse by the square of a right line incommensurable to the greater in length, then

IV. If the greater name be commensurable to the propounded rationall line in length, it is called a fourth binomiall.

V. If the lesser name be so, a fift.

VI. If neither, a sixt.

PROP. XLIX.



To find out a first binomiall line, EG.

a sch.29.10. b 2.lem.10. 10. c 3.lem.10. 3 10.

a Take AB, AC, square numbers, whose excesse CB is not Q. let D be propounded ρ'. b Take EF ⊓⊔ D, and c make AB.CB :: EFq.FGq. then EG shall be a 1 bin.

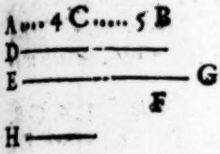
d constr. e 6.def.10. f 6.10. g sch.12.10. h 9.10. k 9.10. l 1.def.48. 10.

For EF d ⊓⊔ D. e therefore EF is ρ'. f also EFq ⊓⊔ FGq. g therefore FG is also ρ'. likewise because EFq. FGq :: AB.CB :: Q.not Q. h therefore EF ⊓⊔ FG. Lastly because by conversion of proportion, EFq. EFq − FGq :: AB. AC :: Q.Q. thence EF k shall be ⊓⊔ √ EFq − FGq. l therefore EG is a 1 binomiall. *W.W. to be Done.*

In numbers thus; let there be D 8. EF 6. AB 9. CB 5. wherefore because 9.5 :: 36. 20, therefore FG is √ 20. and consequently EG is 6 + √ 20.

PROP.

PROP. L.



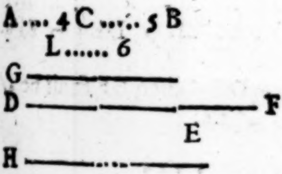
To find out a second binomiall line, EG.

Take AB and AC square numbers, the excesse of which is CB not Q. Let D be the line propounded ρ. take FG $\sqcap\sqcup$ D, and make CB. AB :: FGq. EFq. then EG will be the line desired. *Prove it as the prec.*

For FG $\sqcap\sqcup$ D. wherefore FG is ρ. Also EFq $\sqcap\sqcup$ FGq. therefore EF is ρ. Likewise because FGq. EFq :: CB. AB :: not Q. Q. thence FG is $\sqcap\sqcup$ EF. Lastly seeing CB. AB :: FGq. EFq, and inversely AB. CB :: EFq. FGq. therefore as in in the foregoing *prop.* EF $\sqcap\sqcup$ $\sqrt{}$ EFq $-$ FGq. *a* whereby EG is a 2. binomiall. *W. W. to be Done.* 22. *def.* 48. 10.

In numbers; let there be D 8, FG 10, AB 9, CB 5, then EF is $\sqrt{}$ 180. wherefore EG is 10 $+$ $\sqrt{}$ 180.

PROP. LI.



To find out a third binomiall line, DF.

a Take AB, AC, square numbers, the excesse of which CB is not Q. and let L be a number not Q next greater then CB, *viz.* by a unite or two. Let G be the line propounded ρ. *b* Make L. AB :: Gq. DEq. *b* and AB. CB :: DEq. EFq. then DF shall be a 3. bin. *a sch.* 19. 10. *b* 3. *lem.* 10. 10.

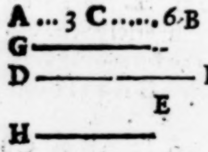
For because DEq *c* $\sqcap\sqcup$ Gq, *d* DE is ρ. also Gq. DEq :: L. AB :: not Q. Q. *e* therefore G $\sqcap\sqcup$ DE. Likewise being that DEq *e* $\sqcap\sqcup$ EFq, *d* also EF is ρ. Moreover because DEq. EFq :: AB. CB :: Q not *c constr.* 6. 10. *d sch.* 12. 10. *e* 6. 10.

f 9.10. g sch.27.8. h 9.10. Q. f is DE ⊔⊓ EF. and being that by constr. and of equality Gq.EFq :: L.CB :: not Q.Q. (for g L and CB are not like plane numbers.) h therefore shall G be also ⊔⊓ EF. Lastly, as in the prec. *prop.* $\surd$ DEq − EFq ⊔⊓ DE. k therefore DF is a 3 bin. *Which was to be Done.*

k 3.def.48.10.

In numbers; let there be AB, 9. CB, 5. L, 6. G, 8. then shall be DE $\surd\,96$, & EF $\surd\,\frac{480}{9}$. wherefore DF $= \surd\,96 + \surd\,\frac{480}{9}$.

PROP. LII.



To find out a fourth binomiall line DF.

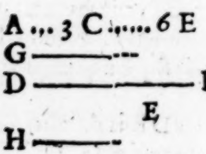
a sch.29.10 a Take any square number AB, and divide it into AC, CB not squares. Let G be the line propounded. b take DE ⊔⊓ G. c and make AB. CB :: DEq. EFq. then DF shall be a 4. bin.

b 1 lem.10.10. c 3.lem.10.10. d 9.10. e 4.def.48.10.

For, as in the 49 of this Book, DF may be shewn to be a binom. and also because by constr. and conversion of proportion DEq.DEq − EFq :: AB.AC :: Q not Q. d shall DE be ⊔⊓ $\surd$ DEq − EFq. e therefore DF is a 4.bin.

In numbers, let G be 8, DE, 6. then EF shall be $\surd$ 24. therefore DF is $6 + \surd\,24$.

PROP. LIII.



To find out a fift binomiall line DF.

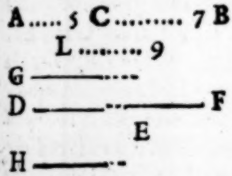
Take any square number AB, whose segments AC, CB are not Q. Let G be the line propounded. take EF ⊔⊓ G. and make CB.AB :: EFq.DEq. then shall DF be a 5.bin.

For

For DF shall be a bin. as in the 50. of this book. and because by construction, and inversion, DEq. EFq :: AB. CB and so by conversion of proportion, DEq. DEq — EFq :: AB. AC :: Q. not Q. a therefore shall DE be ⊏⊐ √ DEq — EFq. b therefore DF is 5 bin. *W.W. to be Done.* a 9.10. b 5.def. 48. 10.

In numbers, let there be G, 7. EF, 6. then DE shall be √ 54. wherefore DF is 6 + √ 54.

Prop. LIV.



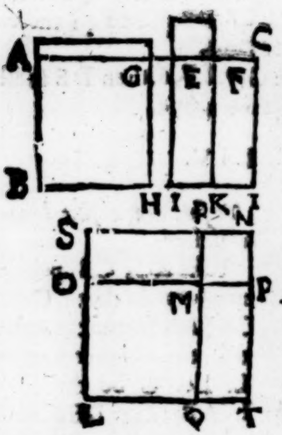
To find out a sixt binomiall line.

Take AC, CB, prime numbers, so that AC + CB (AB) be not Q. take also any number square L. Let G be the line propounded ρ. a and make L. AB :: Gq. DEq. and AB. CB :: DEq. EFq. then DF shall be a 6. binomiall. a 3.lem. 10. 10.

For DF may be demonstrated bin. as in the 51. of this Book. and also by reason that DE and EF ⊏⊐ G. lastly likewise because by constr. and conversion of proportion DEq. DEq — EFq :: AB. AC :: not Q. Q. (For AB is prime to AC, b and so unlike to it) c therefore DE ⊏⊐ √ DEq — EFq. d therefore DF is a 6. bin. *Which was required.* b sch. 27. 8. c 9. 10 d 6. def. 48. 10.

In numbers, let there be G 6. DE √ 48. then EF shall be √ 28. wherefore DF is √ 48 + √ 28.

Lemma.



Let AD *be a rectangle, and the side thereof* AC *divided unequally in* E; *also let the lesser portion* EC *be equally divided in* F. *upon the line* AE [a] *make the rectangle* AGE = EFq. *and from the points* G, E, F [b] *draw* GH, EI, FK, *parallel to* AB. [c] *Let the square* LM *be made, equall to the rectangle* AH, *and upon* OMP *produced the square* MN = GI. *& let the right lines* LOS, LQT, NRS, NPT *be produced.*

a 28.6.

b 31.1.
c 14.2.

I say 1. MS, MT, *are rectangles.* For by reason of the right angles of the squares OMQ, RMP, [a] shall QMR be a right line. [b] therefore RMO, QEP, are right angles. wherefore the parallelograms MS, MT are rectangles.

a sch.15.1.
b 13.2.

2. *Hence it is plain that* LS [c] = LT, *and consequently that* LN *is a square.*

c 2.ax.1.

3. *The rectangles* SM, MT, EK, FD *are equall.* For because the rectangle AGE [d] = EFq. [e] thence shall AE.EF :: EF.GE. [f] and so AH.EK :: EK.GI. that is by constr. LM.EK :: EK. MN. [g] but LM. SM :: SM. MN. therefore EK [h] = SM [k] = FD [l] = MT.

4. *Hence* LN [m] = AD.

5. *Being that* EC *is equally divided in* F, *it is plain that* EF, FC, EC *are* ⊓⊔.

6. If AE ⊓⊔ EC, and AE ⊓⊔ √ AEq − ECq, [o] then shall AG, GE, AE, be ⊓⊔. also, because AG. GE :: AH. GI. [p] therefore shall AH, GI, *i. e.* LM, MN, be ⊓⊔. Likewise thereupon,

d hyp.
e 17.6.
f 1.6.
g sch.22.6.
h 9.5.
k 36.1.
l 43.1.
m 2.ax.1.
n 16.10.

o 18. and 26. 10.
p 10. 10.

7. OM

7. OM ⊔ MP. For by the Hyp. AE ⊔ EC. *p* therefore EC ⊔ GE. *q* wherefore EF ⊔ GE. q 14. 10.
but EF. GE :: EK. GI. *r* therefore EK ⊔ GI, that r 10. 10.
is, SM ⊔ MN. but SM. MN :: OM. MP. *r* therefore OM ⊔ MP.

8. If AE be supposed ⊔ $\surd$ AEq − ECq, it is apparent that AG, GE, AE, are ⊔. whence LM ⊔ MN. for AG. GE :: AH. GI :: LM. MN. fig, and 17. 10.

These being well considered, we shall easily dispatch the six following Propositions.

PROP. LV.

If a space AD *be conteined under a rationall line* AB, *and a first binomiall line* AC (AE + EC) *the right line* OP *which conteineth that space in power is irrationall, and called a binomiall line.*

All that being supposed which is described and demonstrated in the next foregoing *Lemma*, it is manifest that the right line OP containeth in power the space AD. *a* Likewise AG, GE, AE, are ⊔. therefore seeing AE *b* is ρ' ⊔ AB. *c* shall also AG and GE be ρ' ⊔ AB. *d* therefore the rectangles AH, GI, that is, the squares LM, MN are ρ'a. therefore OM, MP are ρ' *e* $\not\sqcup$. *f* and consequently OP is a binomiall. *W. W. to be Dem.*

a hyp. and lem. 54. 10.
b hyp.
c sch. 12. 10.
d 20. 10.
e lem. 54. 10.
f 37. 10.

In numbers, let there be AB 5. AC 4 + $\surd$ 12. wherefore the rectangle AD = 20 + $\surd$ 300 = to the square LN. therefore OP is $\surd$ 15 + $\surd$ 5. namely a 6 binomial.

PROP. LVI.

If a space AD *be comprehended under a rationall line* AB, *and a second binomiall* AC (AE + EC) *the right line* OP, *which containeth that space* AD *in power, is irrationall, and called a first mediall line.*

The

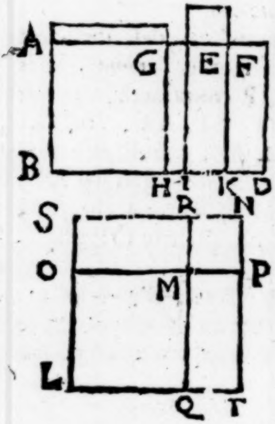
The foresaid *Lemma* of the 54. of this Book being again supposed, then shall OP be = √ AD. *a* also AE, G, GE are ⊐⊏. therefore since AE *b* is ρ' ⊐⊏. AB, likewise AG, GE *c* shall be ρ' ⊐⊏ AB. therefore the rectangles AH, GI, *i.e.* OMq, MPq. *d* are μα. *e* Moreover OM ⊐⊏̸ MP. Lastly EF ⊐⊏ EC, and EC *f* ⊐⊏ AB, *g* wherefore EK, *i.e.* SM, or OMP, is ρν. *h* Consequently OP is a first bimediall. *Which was to be Dem.*

a hyp. and lem. 54. 10.
b hyp.
c sch. 12. 10.
e lem. 54. 10.
f hyp. 12. 10.
g 20. 10.
h 38. 10.

In numbers, let there AB, 5; and AC, √ 48 : + 6. then the rectangle AD = √ : 1200 + 30 = OPq. therefore OP is *v*√ 675 + *v*√ 75. *viz.* a first bimedial.

See Scheme 57.

PROP. LVII.



If a space AD *be contained under a rationall line* AB *and a third binomiall line* AC (AE + EC) *the right line* OP *which containeth in power the space* AD, *is irrationall, and called a second bimediall line.*

As above, OPq = AD. also the rectangles AH, GI, that is OMP, MPq are μα. *a* Likewise EK or OMP is μν. *b* therefore OP is a second bimediall.

a hyp. and 22. 10.
b 39. 10.

In numbers; let there be AB 5, AC √ 32 + √ 34. wherefore AD is √ 800 + √ 600 = OPq. and so OP is *v*√ 450 + *v*√ 50. that is a 2. bimed.

PROP.

PROP. LVIII.

If a space AD *be comprehended under a rationall line* B *and a fourth binomiall* AC (AE + EC) *the right line* OP *containing the space* AD *in power, is that irrationall line which is called a Major line.*

For again, OMq [a] ⊐⊏ MPq; & the rectangle AI, *i e.* OMq + MPq [b] is $\rho\nu$. [c] also EK or OMP is $\mu\nu$. [d] therefore OP ($\sqrt{}$AD) is a *Major* line. *W.W. to be Dem.*

a lem.54. 10.
b hyp. and 20.10
c hyp and 22.10.
d 40. 10.

In numbers; let there be AB 5. and AC 4 + $\sqrt{}$ 8. then the rectangle AD is 20 + $\sqrt{}$ 200. wherefore OP is $\sqrt{}$: 20 + $\sqrt{}$ 200.

PROP. LIX.

If a space AD *be contained under a rationall line* AB, *and a fift binomiall* AC, *the right line* OP *which containeth the space* AD *in power, is that irrationall line, which is a line containing a rationall and a mediall rectangle in power.*

Again OMP ⊐⊏ MPq. and the rectangle AI or OMq + MPq is $\mu\nu$. [a] Likewise the rectangle EK or OMP is $\rho\nu$. [b] therefore OP ($\sqrt{}$ AD) contains in power $\rho\nu$ and $\mu\nu$. *W.W. to be Dem.*

a as in the prec.
b 41. 10.

In numbers, let there be AB 5. and AC 2 + $\sqrt{}$ 8. then the rectangle AD = 10 + $\sqrt{}$ 200 = OPq. Wherefore OP is $\sqrt{}$: 10 + 200.

PROP.

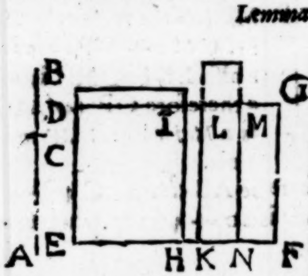
PROP. LX.

If a space AD *be contained under a rationall line* AB *and a sixt binomiall* AC (AE + EC) *the line* OP *containing the space* AD *in power is irrationall, which containeth in power two mediall rectangles.*

As often before, OMq ⊤⊔ MPq, and OMq + MPq is $\mu\nu$. and also the rectangle (EK) OMP is $\mu\nu$.
a 41. 10. *a* therefore OP = √ AD contains in power 2 $\mu\alpha$. *W.W. to be Dem.*

In numbers, let there be AB 5. AC √ 12 + √ 8. therefore the rectangle AD or OPq is √ 300 + √ 200. and so OP is √ : √ 300 + √ 200.

Lemma.



Let a right line AB *be unequally divided in* C, *and let* AC *be the greater portion. and upon some line* DE *apply the rectangles* DF = ABq, *and* DH = ACq, *and* IK = CBq. *and let* LG *be divided equally in* M, *and also* MN *drawn parallel to* GF.

I say 1. The rectangle ACB is = LN or MF. *a* For
a 4. 2. and 3. ax. 1. 2 ACB = LF.
b 7. 2. 2. DL ⊏ LG. for DK (ACq + CBq) *b* ⊏ LF (2 ACB) therefore being DK, LF are of equall al-
c 1. 6. titude, *c* DL shall be ⊏ LG.
d 16. 10. 3. If AC ⊤⊥ CB, *d* then shall the rectangle DK be ⊤⊔ ACq and CBq.

4. *Also* DL ⊤⊔ LG. for ACq + CBq *e* ⊤⊔ 2
e lem. 26. 10. ACB, *i. e.* DK ⊤⊔ LF. but DK. LF *c* :: DL. LG.
f 10. 10. *f* therefore DL ⊤⊔ LG.

5. *Moreover* DL ⊤⊔ √ DLq − LGq. For ACq.
g 1. 6. ACB *g* :: ACB. CBq. that is, DH. LN :: LN. IK.
h 17. 6. *c* wherefore DI. LM :: LM. IL. *h* therefore DI × IL
k hyp. = LMq. therefore seeing ACq *k* ⊤⊔ CBq, that is
l 10. 10. DH ⊤⊔ IK. and *l* so DI ⊤⊔ IL. *m* shall DL be ⊤⊔
m 18. 10. √ DLq − LGq. *W.W. to be Dem.*

6. *But*

6. *But if* ACq *be put* ⊐⊏ CBq, [a] *then shall* DL *be* ⊐⊏ $\surd$ DLq − LGq. a 19.10.

This Lemma is preparatory to the 6 following Propositions.

PROP. LXI.

The square of a binomiall line (AC + CB) *applyed unto a rationall line* DE, *makes the latitude* DG *a first binomiall line.*

Those things being supposed, which are described and demonstrated in the next preceding *Lemma*; because AC, CB [a] are ρ' ⊐⊏, [b] the rectangle DK shall be ⊐⊏ ACq. [c] and so DK is ρ'ν. [d] therefore DL ⊐⊏ DE ρ'. but the rectangle ACB, and so 2 ACB (LF) [e] is μν. [f] therefore the latitude LG is ρ' ⊐⊏ DE. [g] therefore also DL ⊐⊏ LG also DL ⊐⊏ $\surd$ DLq − LGq. from whence [k] it follows that DG is a first binomiall. *W.W. to be Dem.*

a hyp.
b lem. 60. 10
c sch. 12.10.
d 21. 10.
e 22. & 24. 10.
f 23. 10.
g 13. 10.
h lem. 60. 10.
k 1 def. 48. 10.

PROP. LXII.

The square of a first bimediall line (AC + CB) *being applied to a rationall line* DE, *makes the latitude* DG *a second binomiall line.*

The aforesaid *Lemma* being again supposed; The rectangle DK ⊐⊏ ACq. [a] therefore DK is μν. [b] therefore the latitude DL is ρ' ⊐⊏ DE. But because the rectangle ACB, and so LF (2 ACB) [c] is ρ'ν, [d] shall LG be ρ' ⊐⊏ DE. [e] therefore DL, LG are ⊐⊏, [f] also DL ⊐⊏ $\surd$ DLq − LGq. [g] from whence it is clear that DG is a second binomiall. *W.W. to be Dem.*

a 24. 10.
b 23. 10.
c hyp. and sch. 12. 10.
d 21. 10
e 13. 10.
f lem 60. 10.
g 2 def. 48 10.

PROP.

PROP. LXIII.

The square of a second bimediall line (AC + CB) *applied to a rationall line* DE *makes the breadth* DG *a third binomiall line.*

a hyp. and 24.10. b 23. 10. c lem.60 10 d 3 def.48. 10.

As in the *prec.* DL is ρ' ⊐⊏ DE. Furthermore because the rectangle ACB, and so LF (2 ACB) a is μν. b therefore shall LG be ρ' ⊐⊏ DE. c Moreover DL ⊐⊏ LG. and also DL ⊐⊏ √DLq − LGq. d therefore DG is a third binomiall. *W.W. to be Dem.*

PROP. LXIV.

The square of a Major line (AC + CB) *applied to a rationall line* DE, *makes the breadth* DG *a fourth binomiall line.*

a hyp. and sch 12.10. b 21.10. c hyp and 24.10. d 23.10. e 13.10. f lem.60.10. g 4.def. 48. 10.

Again ACq + CBq. *i.e.* DK a is ρν. b therefore DL is ρ' ⊐⊏ DE. also ACB, and so LF (2 ACB) c is μν. d therefore LG is ρ' ⊐⊏ DE. e and consequently DL ⊐⊏ LG. Lastly because AC ⊐⊓ BC. f shall DL be ⊐⊏ DLq − LGq. g whence DG is a fourth binomiall. *W.W. to be Dem.*

PROP. LXV.

The square of a line containing in power a rational and a mediall rectangle (AC + CB) *applied to a rationall line* DE *makes the latitude* DG *a fift binomiall.*

a 23.10. b 21. 10. c 13.10. d lem.60. 10. e 5.def.48. 10.

Again, DK is μν. a therefore DL is ρ' ⊐⊏ DE. also LF is ρν. b therefore LG is ρ' ⊐⊏ DE. c therefore DL ⊐⊏ LG. d likewise DL ⊐⊏ √DLq − LGq. e and so by consequence DG is a fift binomiall. *Which was to be Dem.*

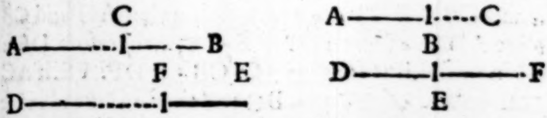
PROP. LXVI.

The square of a line containing in power two mediall rectangles (AC + CB) *applied to a rationall line* DE, *makes the latitude* DG *a sixt binomiall line.*

As

As before, DL and LG and LG are ^{p} ⊓⊔ DE. a hyp.
But for that ACq + CBq (DK) ^{a} ⊓⊔ ACB, b 14.10.
^{b} and so DK ⊓⊔ LF (2 ACB) and also DK. LF ^{c} :: c 1.6.
DL. LG. ^{d} therefore shall DL be ⊓⊔ LG. ^{e} Lastly d 10.10.
DL ⊓⊔ $\surd$ DLq − LGq. ^{f} by which it appears that e lem. 60. 10.
DG is a sixt binomiall. f 6 def. 48. 10.

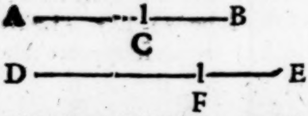
Lemma.



Let AB, DE *be* ⊓⊔ *and make* AB. DE :: AC. DF.
I say 1. AC ⊓⊔ DF. as appears by 10. 10. also a 19. 5.
CB ⊓⊔ FE. ^{a} because AB. DE :: CB. FE.
2. AC. CB :: DF. FE. For AC. DF :: AB. DE
:: CB. FE. therefore inversely AC. CB :: DF.
FE.
3. *The Rectangle* ACB ⊓⊔ DFE. For ACq. ACB
^{b} :: AC. CB ^{c} :: DF. EF :: DFq. DFE. wherefore by b 1. 6.
inversion ACq. DFq :: ACB. DFE. therefore being c before.
ACq ⊓⊔ DFq. ^{d} shall ACB be ⊓⊔ DFE. d 10. 10.
4. ACq + CBq ⊓⊔ DFq + FEq. For because
ACq. CBq ^{e} :: DFq. FEq. therefore by addition e 22. 6.
ACq + CBq. CBq :: DFq + FEq. FEq. therefore
being CBq ⊓⊔ FEq, ^{f} shall also ACq + CBq be f 10. 10.
⊓⊔ DFq + FEq.
5. *Hence,* If AC ⊓⊢ or ⊓⊢ CB, ^{g} then likewise g 10. 10.
shall DE be ⊓⊢ or ⊓⊢ EF.

PROP.

Prop. LXVII.



A line DE, *commensurable in length to a binomiall line* (AC + CB) *is it self a binomiall line, and of the same order.*

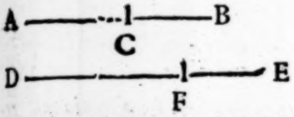
Make AB. DE :: AC.DF.[a] then are AC, DF ⊐⊏.
a lem.66.10. [a] and CB, FE ⊐⊏. whence being that AC and CB
b hyp. [b] are ρ' ⊐⊏̲, [c] thence DF, FE ρ' ⊐⊏̲. therefore DE is
c lem.66.10. and sch.12.10. a binomiall. But for that AC. CB [a] :: DF. FE. if AC
⊐⊏ or ⊐⊏̲ √ ACq − BCq, [d] then in like manner
d 15. 10. DF ⊐⊏ or ⊐⊏̲ √ DFq − FEq. also if AC ⊐⊏ or
e 12.10. and 14.10. ⊐⊏ ρ' propounded, [e] then shall DF be ⊐⊏ or ⊐⊏ ρ'
propounded. But if CB ⊐⊏ or ⊐⊏ ρ'. likewise FE
g 14.10. ⊐⊏ or ⊐⊏ ρ'. If both AC, CB, ⊐⊏ ρ'. [f] then also
f by def. 48. 10. both DF, FE, ⊐⊏ ρ'. [g] That is, whatsoever binomiall
AB is, DE shall be of the same order. *W.W. to be Dem.*

Prop. LXVIII.

A line DE *commensurable in length to a bimediall line* (AC + CB) *is also a bimediall line, and of the same order.*

a 12.6. Make AB. DE :: AC, DF. [b] therefore AC ⊐⊏
b lem.66. 10. DF. and CB ⊐⊏ FE. therefore seeing AC and CB
c hyp. [c] are μ, [d] also DF and FE shall be μ. and for that AC
d 24.10. [e] ⊐⊏̲ CB, [e] therefore FD ⊐⊏̲ FE. [f] therefore DE is
e 10.10.
f 38.10. 2 μ. Wherefore if the rectangle ACB be $\rho'\nu$. because
g sch.12. 10. DFE [h] ⊐⊏ ACB, [g] likewise DFE is $\rho'\nu$. and if that
h 24.10. be $\mu\nu$, [h] this shall be $\mu\nu$ too. [k] That is, whether AB be
k 38. or 39. 10. 1 bimed. or 2 bimed. DF shall be of the same order.
W.W. to be Dem.

PROP. LXIX.



A line DE *commensurable to a Major line* (AC+CB) *is it self a Major line.*

Make AB.DE :: AC.DF. Because AC [a] ⊓⊔ CB, [b] thence DF ⊓⊔ FE. also ACq + CBq [a] is ρ̀ν. and so being DFq + FEq [b] ⊔⊓ ACq + CBq , [c] also DFq + FEq is ρ̀ν. lastly, the rectangle ACB [a] is μν. [d] therefore the rectangle DFE is μν. (because DFE is [b] ⊔⊓ ACB) [e] wherefore DE is *a Major* line. *W.W. to be Dem.*

a hyp. b lem. 66.10 c sch. 12. 10. d 24.10. e 40.10.

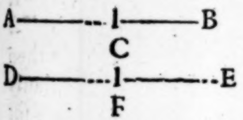
PROP. LXX.

A line DE *commensurable to a line containing in power a rationall and a mediall rectangle* (AC+CB) *is a line containing in power a rationall and a mediall rectangle.*

Again make AB. DE :: AC. DF. Because AC [a] ⊓⊔ CB, [b] also DF ⊓⊔ FE. likewise because ACq + CBq [a] is μν, [c] therefore DFq + FEq shall be μν. lastly because the rectangle ACB [a] is ρ̀ν, [d] also DFE is ρ̀ν. Therefore DE contains in power ρ̀ν and μν. *W.W. to be Dem.*

a hyp. b lem. 66. 10. c 24 10. d sch. 12. 10. e 41. 10.

PROP. LXXI.



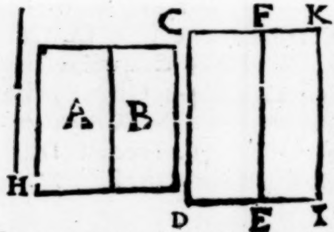
A line DE *commensurable to a line containing two mediall rectangles in power* (AC + CB) *is also a line containing in power two mediall rectangles.*

Divide DE, as in the prec. Because ACq [a] ⊔⊓ CBq, [b] thence shall DFq be ⊔⊓ FEq. also for that ACq + CBq [a] is μν, [c] shall DFq + FEq be also μν. And in like manner because ACB [a] is μν, [d] also DFE is μν. Lastly, because ACq + CBq ⊔⊓ ACB, [e] shall

a hyp. b lem. 66. 10 c 24. 10. d 24. 10.

e 14. 10. *e* shall DFq + FEq be ⌐⊓⌐ DFE. *f* From whence
f 42. 10. it follows that DE contains in power 2 $\mu\alpha$. *Which was to be Dem.*

PROP. LXXII.

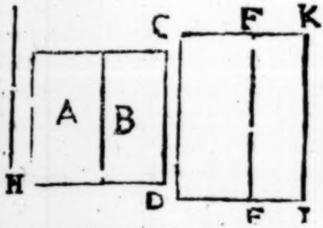


If a rationall rectangle A *and a mediall* B *, be composed together, these four irrationall lines will be made ; either a binomiall, or a first bimediall, or a major, or a line containing in power a rationall and a mediall rectangle.*

Namely, if Hq = A + B, then H shall be one of the four lines which the Theoreme mentions. For
a cor. 16. 6. upon CD the propounded ρ', *a* make the rectangle
b 2. ax. 1. CE = A, and FI = B. *b* and so CI = Hq. Whereas
c 21. 10. then is A $\rho' r$, likewise CE is $\rho' r$. *c* therefore the latitude CF is ρ' ⌐⊓⌐ CD. and because B is μr, also FI shall
d 23. 10. be μr. *d* therefore FK is ρ' ⌐⊓⌐ CD. *e* therefore CF,
e 13. 10. FK are ρ' ⌐⊓⌐. and so the whole CK *f* is binom.
f 37. 10. wherefore if A ⊏ B, *i.e.* CE ⊏ FI, *g* then CF ⊏
g 1. 6. FK. therefore if CF ⌐⊓⌐ $\sqrt{}$ CFq − FKq, *h* likewise
h 1. def. 48. 10 CK shall be a 1. bin. and consequently H = $\sqrt{}$ CI
k 55. 10. *k* is a bin. If CF be supposed ⌐⊓⌐ $\sqrt{}$ CFq − FKq,
l 4 def. 48. 10. *l* then shall CK be a 4. bin. wherefore H ($\sqrt{}$ CI) *m* is
m 58. 10. a major line. But if A ⊐ B, *g* then shall CF be ⊐
n 2. def. 48. 10. FK. consequently if FK ⌐⊓⌐ $\sqrt{}$ FKq − CFq, *n* then
o 56. 10. shall CK be a 2. bin. *o* wherefore H is a first 2 μ. lastly
p 5. def. 48. 10. if FK ⌐⊓⌐ $\sqrt{}$ FKq − CFq, *p* then CK shall be a fift
q 59. 10. binom. *q* whence H shall contein in power $\rho' r$ and μr. *W. W. to be Dem.*

PROP.

PROP. LXXIII.



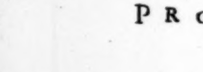
If two mediall rectangles A, B, *incommensurable to one another be composed together, the two remaining irrationall lines are made, either a second bimediall, or a line containing in power two mediall rectangles.*

As H containing in power A + B is one of the said irrationall lines. For upon CD propounded ρ' draw the rectangle CE = A, and FI = B. whence Hq = CI. Therefore because CE and FI *a* are $\mu\alpha$. *b* the latitudes CF, FK, shall be ρ' $\sqcup$ CD. also because CE *a* $\sqcup$ FI, and CE.FI *c* :: CF.FK, *d* therefore CF $\sqcup$ FK. *e* therefore CK is a 3. bin. namely, if CF $\sqcup$ $\surd$ CFq − FKq. whence H = $\surd$ CI *f* shall be a second 2 μ. But if CF $\sqcup$ $\surd$ CFq − FKq, *g* then CK shall be a 6 binom. *h* and consequently H conteins in power 2 $\mu\alpha$. *W. W. to be Dem.*

a *hyp.*
b 23. 10.
c 1. 6
d 10. 10
e 3. *def.* 48. 10.
f 57. 10.
g 6. *def* 48. 10
h 60. 10.

Here begin the Senaries of lines irrationall by Subtraction.

PROP. LXXIV.



If from a rationall line DF *a rationall line* DE, *commensurable in power only to the whole* DF, *be taken away, the residue* EF *is irrationall, and is called an Apotome or residuall line.*

For EFq *a* $\sqcup$ DEq; *b* but DEq is $\rho'v$; *c* therefore EF is ρ'. *W. W. to be Dem.*

In numbers; let there be DF, 2. DE, $\surd$ 3. then EF shall be 2 − $\surd$ 3.

a *lem.* 26. 10
b *hyp*
c 10. & 11. *def.* 10.

Prop. LXXV.

D E F

If from a mediall line DF, *a mediall line* DE *commensurable only in power to the whole* DF, *and comprehending with the whole* DF *a rationall rectangle, be taken away, the remainder* EF *is irrationall, and is called a first residuall line of a mediall.*

a sch.16 10. For EFq *a* $\sqcap\!\sqcup$ to the rectangle FDE. therefore
b hyp. seeing FDE *b* is ρ'ν, *c* EF shall be ρ'. *W.W. to be Dem.*
c 20. and 11. def. 10. In numbers, let DF be $v\sqrt{54}$, and DE $v\sqrt{24}$. therefore EF is $v\sqrt{54} - v\sqrt{24}$.

Prop. LXXVI.

D E F

If from a mediall line DF, *a mediall line* DE *be taken away being commensurable only in power to the whole* DF, *and comprehending together with the whole line* DF *a mediall rectangle, the remainder* EF *is irrationall, and is called a second residuall of a mediall line.*

a hyp. Because DFq and DEq *a* are μν $\sqcap\!\sqcup$, *b* therefore
b 16. 10. shall DFq + DEq be $\sqcap\!\sqcup$ DEq. *c* wherefore DFq +
c 24. 10. DEq is μν. also the rectangle FDE, *c* and so 2 FDE,
d cor. 7. 2. *a* is μν. therefore EFq (*d* DFq + DEq − 2 FDE)
e 27. 10. *e* is ρ'ν. wherefore EF is ρ'. *W.W. to be Dem.*

In numbers, let DF be $v\sqrt{18}$. and DE $v\sqrt{8}$. then EF $v\sqrt{18} - v\sqrt{8}$.

Prop. LXXVII.

A B C

If from a right line AC *be taken away a right line* AB *being incommensurable in power to the whole* BC, *and making with the whole* AC *that which is composed of their squares rationall, and the rectangle contained under them mediall, the remainder* BC *is irrationall, and is called a Minor line.*

a hyp. For ACq + ABq *a* is ρ'ν. but the rectangle ACB
b sch. 12. 10. *a* is μν. *b* therefore 2 CAB $\sqcap\!\sqcup$ ACq + ABq (2 *c* CAB
c 7. 2. +

$+ BCq.)$ [d] therefore $ACq + ABq$ $\sqsupset\!\sqsubset$ $BCq.$ [e] therefore BC is $\dot{\rho}$. *W.W. to be Dem.* d 17. 10. e 11. def. 10.

In numbers, let AC be $\sqrt{:18 + \sqrt{108}}$; AB $\sqrt{:18 - \sqrt{108}}$. then BC is $\sqrt{:18 - \sqrt{:108}} - \sqrt{:18 - \sqrt{108}}$.

PROP. LXXVIII.

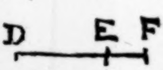
D———E———F *If from a right line* DF *be taken away a right line* DE, *being incommensurable in power to the whole line* DF, *and with the whole* DF *making that which is composed of their squares mediall, and the rectangle contained under the same lines rationall, the line remaining* EF *is irrationall, and is called a line making a whole space mediall with a rationall space.*

For 2 FDE [a] is $\rho\nu$. [b] and $DFq + DEq$ is $\mu\nu$. [c] therefore 2 FDE $\sqsupset\!\sqsubset$ $DFq + DEq$ [d] $(2 FDE + EFq)$ [e] therefore EF is $\dot{\rho}$. *W.W. to be Dem.*

a hyp. & sch. 12. 10. b hyp. c sch. 12. 10. d 7. 2. e sch. 12. 10. and 11. def. 10.

In numbers, let DF be $\sqrt{:\sqrt{216} + \sqrt{72}}$; DE $\sqrt{:\sqrt{216} - \sqrt{72}}$. therefore EF is $\sqrt{:\sqrt{216} + \sqrt{72}} - \sqrt{:\sqrt{216} - \sqrt{72}}$.

PROP. LXXIX.



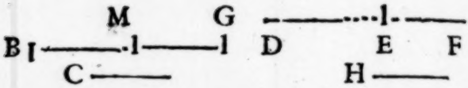
If from a right line DF *be taken away a right line* DE, *incommensurable in power to the whole* DF, *and which together with the whole makes that which is composed of their squares mediall, and the rectangle contained under them also mediall and incommensurable to that which is composed of their squares, the remainder is irrationall, and is called a line making a whole space mediall with a mediall space.*

For 2 FDE, and $FDq + DEq$ [a] are $\mu\alpha$; [b] therefore EFq ([c] $DFq + DEq - 2 FDE$) is $\rho\nu$. [d] and so consequently EF is $\dot{\rho}$. *W. W. to be Dem.*

a hyp. & 24. 10. b 27. 10. c cor. 7. 2. d 11. def. 10.

In numbers; let DF be $\sqrt{:\sqrt{180} + \sqrt{60}}$. DE $\sqrt{:\sqrt{180} - \sqrt{60}}$. then EF shall be $\sqrt{:\sqrt{180} + \sqrt{60}} - \sqrt{:\sqrt{180} - \sqrt{60}}$.

Lemma.



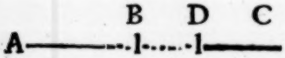
If there be the same excesse between the first magnitude BG *and the second* C (MG) *as is between the third magnitude* DF *and the fourth* H(EF;) *then alternately, the same excesse shall be between the first magnitude* BG *and the third* DF , *as is between the second* C *and the fourth* H.

a hyp. For because that [a] to the equalls BM, DE, are added the equalls MG, EF, that is, C, H; the excesse of
b 15.ax.1. the wholes BG, DF, [b] shall be equall to the excesse of the parts added C, H. *W.W. to be Dem.*

Coroll.

Hence, Four magnitudes Arithmetically proportionall, are alternately also Arithmetically proportionall.

P R O P. LXXX.

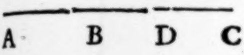


To an Apotome or residuall line AB *onely one rationall right line* BC, *being commensurable in power onely to the whole* AB, *is congruent , or can be joyned.*

If it be possible, let some other line BD be added
a 22. 10. to it; [a] then the rectangles ACB, ADB, [b] and so con-
b 22. 10. sequently the doubles of them are μx. wherefore
c cor. 7 2. seeing $ACq + BCq - 2\,ACB$ [c] $= ABq$ [c] $= ADq$
d lem. 79. 10. $+ DBq - 2\,ADB$. therefore alternately $AGq +$
e hyp. and 27. 10. $BCq -: ADq + BDq$ [d] $= 2\,ACB -: 2\,ADB$.
f sch. 12. 10. But $ACq + BCq -: ADq + BDq$ [e] is ρ'ν. [f] therefore
g 27. 10. $2\,ACB -: 2\,ADB$ is ρ'ν. [g] *Which is Absurd.*

PROP.

PROP. LXXXI.

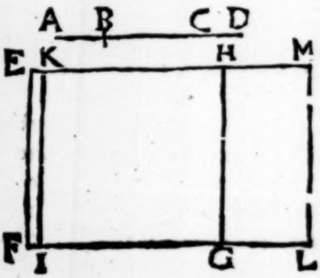


To a first mediall residuall line AB *onely one mediall right line* BC, *being commensurable only in power to the whole, and comprehending with the whole line a rationall rectangle, can be joyned.*

Conceive BD to be such a line as may be joyned to it; then because ACq and BCq, as well as ADq and BDq [a] are $\mu\alpha$. ⊓⊔. [b] also ACq + BCq, & ADq + BDq shall be $\mu\alpha$. [c] but the rectangles ACB, ADB, [d] and so 2 ACB and 2 ADB are $\rho'\alpha$, [e] therefore 2 ACB − : 2 ADB, [f] that is ACq + BCq − : ADq + BDq is $\rho'\nu$. [g] *Which is Absurd.*

a *hyp.*
b 10 *and* 24. 10.
c *hyp.*
d *sch.* 12. 10.
e *sch.* 27. 10.
f 7. 2. *and lem.* 79. 10.
g 27. 10.

PROP. LXXXII.



Unto a second mediall residuall line AB *onely one mediall right line* BC, *commensurable onely in power to the whole, and with it conteining a mediall rectangle, can be joyned.*

If it be possible, let some other line BD be added to it; and upon EF ρ' make the rectangle EG = ACq + BCq; as also the rectangle EL = ADq + BDq. likewise EI = ABq. Now 2 ACB + ABq = ACq + BCq = EG. therefore seeing EI = ABq, [a] also KG shall be = 2 ACB. moreover ACq and BCq [b] are $\mu\alpha$ ⊓⊔. [c] therefore EG (ACq + BCq) is $\mu\nu$. [d] therefore the breadth EH is ρ' ⊓⊔ EF. [e] Further, the rectangle ACB [f] and so 2 ACB (KG) is $\mu\nu$. [d] therefore KH is also B, ρ' ⊓⊔ EF. lastly, because ACq + BCq (EG) [g] ⊓⊔ 2 ACB (KG) and EG. KG [h] :: EH. KH. [k] therefore

a 4. 2. *and* 3. *ax.* 1.
b *hyp.*
c 24. 10.
d 13. 10.
e *hyp.*
f 24. 10
g *lem.* 26. 10.
h 1. 6
k 10. 10.

 fore

74.10. fore EH ⊓⊔ HK. [l] therefore EK is a residuall line, whereto KH is congruent. by the same reason also shall KM be congruent to the said EK. *which is repugnant to the* 80.*prop. of this Book.*

PROP. LXXXIII.

A B D C

To a Minor line AB *onely one right line* BC *can be joined being incommensurable in power to the whole, and making together with the whole line that which is composed of their squares rationall, and the rectangle which is conteined under them mediall.*

a hyp.
b lem 97. 10.
c sch.27.10.
d 27.10.

Conceive any other BD to be congruent to it; Therefore whereas ACq + BCq, and ADq + BDq [a] are ῥα, their excesse (2 [b] ACB —: 2 ADB) [c] is ῥν. *Which is Absurd;* because ACB and ADB are μα by the Hyp.

PROP. LXXXIV.

A B D C

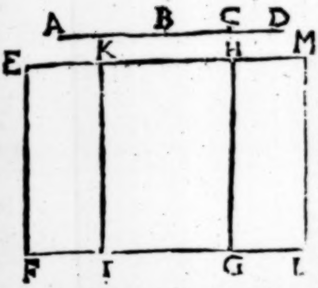
Unto a line (AB) *making with a rationall space a whole space mediall onely one right line* BC *can be joined, being incommensurable in power to the whole, & making together with the whole that which is composed of their squares mediall, and the rectangle which is contained under them rationall.*

a hyp.
b sch. 12.10.
c lem.79. 10.
d sch.27.10.

Suppose some other BD to be congruent also to it; [a] then the rectangles ACB, ADB, [b] and so 2 ACB and 2 ADB are ῥα. therefore 2 ACB —: 2 ADB, [c] that is, ACq + BCq —: ADq + BDq [d] is ῥν. *Which is Absurd.* since ACq + BCq, and ADq + BDq are μα by the Hyp.

PROP.

PROP. LXXXV.



To a line AB, *which with a mediall space makes a whole space mediall, can be joined onely one right line* BC, *incommensurable in power to the whole, and making with the whole both that which is composed of their squares mediall, and the rectangle which is contained under them mediall and incommensurable to that which is composed of their squares.*

Those things being supposed which are done and shewn in the 82. *prop.* of this Book; it is clear that EH and KH are ῥ ⊓ EF. Besides, being that ACq + CBq, that is, the rectangle EG,[a] is ⊓ ACB.[b] & a hyp.
so EG ⊓ 2 ACB (KG;) & EG.KG[c] :: EH.KH; b 14. 10.
shall EH be ⊓ KH. therefore EK is a residuall c 1. 6.
line, & the line congruent to it is KH. In like manner may KM be shewn to be congruent to the said residuall EK, against the 80. *prop.* of this Book.

Third Definitions.

A Rationall line & a residuall being propounded, if the whole be more in power then the line joined to the residuall, by the square of a right line commensurable unto it in length; then

I. If the whole be commensurable in length to the rationall line propounded, it is called a first residuall line.

II. But if the line adjoined be commensurable in length to the rationall line propounded, it is called a second residuall line.

III. If neither the whole nor the line adjoined be commensurable in length to the rationall line propounded, it is called a third residuall line.

Moreover

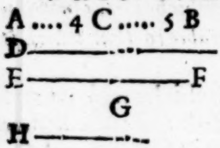
Moreover, If the whole be more in power then the line adjoined by the square of a right line incommensurable to it in length, then

I V. If the whole be commensurable in length to the rationall line propounded, it is called a fourth residuall line.

V. But if the line adjoined be commensurable in length to the rationall line propounded, it is a fift residuall.

V I. If neither the whole nor the line adjoined be commensurable in length to the rationall line propounded, it is termed a sixt residuall line.

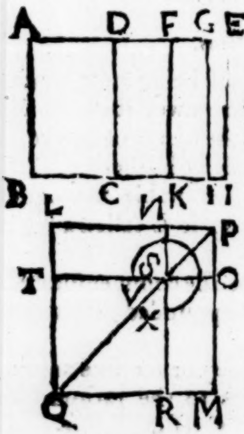
PROP. LXXXVI, 87,88,89,90,91.



To find out a first, second, third, fourth, fift, and sixt residuall line.

Residuall lines are found out by subducting the lesse names or parts of binomialls from the greater. Ex. gr. Let $6 + \sqrt{20}$ be a first binom. then shall $6 - \sqrt{20}$ be a first residuall. So that it is not necessary to repeat more concerning the finding of them out.

Lemma.



Let AC *be a rectangle conteined under the right lines* AB, AD. *Let* AD *be drawn forth to* E, *and* DE *equally divided in* F. *and let the rectangle* AGE *be* = FEq. *& the rectangles* AI, DK, FH, *finished. Then let the square* LM = AH *be made, and the square* NO = GI; *and the lines* NSR, OST, *produced.*

I say 1. the rectangle AI = LM + NO = TOq + SOq. which appears by the constr.

2. The

2. *The rectangle* DK = LO. For because the rectangle AGE *a* = FEq. *b* thence are AG, FE, GE, ∺ *c* and so AH, FI, GI ∺, *a* that is LM, FI, NO, ∺; but LM, LO, NO *d* are ∺; therefore FI = *e* LO *f* = DK = *g* NM.
a constr. b 17.6. c 1.6. d sch.22.6. e 9.5. f 36.1. g 43.1.

3. *Hence*, AC = AI − DK − FI = LM + NO − LO − NM = TR.

4. *h It is manifest that* DF, FE, DE, *are* ⊓⊔. h 16.10.

5. *If* AE ⊤⊥ DE, *and* AE ⊓⊔ √ AEq − DEq, *k then shall* AG, GE, AE *be* ⊓⊔. k 18.and 10.10.

6. *Also, because* AE *l* ⊓⊔ DE, *m thence shall* AE, FE *be* ⊓⊔. *n and so* AI, FI, *that is*, LM + NO *and* LO *are* ⊓⊔.
l hyp. m 13.10. n 1.6.and 10.10.

7. *Because* AG * ⊓⊔ GE, *n shall* AH, GI, *that is* LM, NO *be* ⊓⊔. * before.

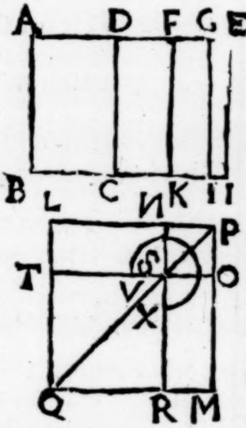
8. *But because* AE *l* ⊓⊔ DE, *o therefore shall* FE, GE *be* ⊓⊔, *n and so the rectangle* FI ⊓⊔ GI, *that is* LO ⊓⊔ NO. *wherefore seeing* LO. NO *p* ∷ TS.SO. *q therefore shall* TS, SO *be* ⊓⊔.
o 14.10. p 2.6. q 10.10.

9. *If* AE *be put* ⊓⊔ √ AEq − DEq, *r then shall* AG, GE, AE *be* ⊓⊔. r 19.10.and 17.10.

10. *s Wherefore the rectangles* AH, GI, *that is* TOq, SOq *shall be* ⊓⊔. s 1.6.and 10.10.

PROP.

PROP. XCII.



If a space AC *be contained under a rationall line* AB, *and a first residuall line* AD (AE — DE) *the right line* TS, *which containeth the space* AC *in power*, *is a residuall line.*

Use the foregoing *Lemma* for a preparatory to the demonstration of this prop. Therefore TS = ✓ AC. Also AG, GE, AE, are ⊐⊏; therefore since AE ⊐⊏ *a* AB ρ', *b* also AG and GE shall be ⊐⊏ AB. *c* therefore the rectangles AH & GI, that is,
a hyp. b 12. 10. c 20. 10.
d lem. 91. 10. e 74. 10. TOq and SOq are ρ'α. *d* Likewise TO, SO, are ρ' ⊐⊢. *e* and consequently TS is a residuall line. *W.W. to be Dem.*

PROP. XCIII.

See the prec. Scheme.

If a space AC *be contained under a rationall line* AB *and a second residuall* AD (AE — DE) *the right line* TS, *containing the space* AC *in power*, *is a first mediall residuall line.*

Again, by the foregoing *Lemma*, A G, GE, AE are
a hyp. ⊐⊏. therefore *a* since AE is ρ' ⊐⊏ AB, *b* also AG,
b 13. 10. GE, shall be ρ' ⊐⊏ AB. *c* therefore the rectangles
c 22. 10.
d lem 74. 10. AH, GI, that is, TOq, SOq are μα. *d* likewise TO
e hyp. ⊐⊢ SO. Lastly, because DE *e* ⊐⊏ AB ρ', *f* the right
f 20. 10. angle DI, and the half thereof DK or LO, that is
g 75. 10. TOS shall be ρ'ν. *g* from whence it follows that TS (✓ AC) is a first mediall residuall. *W. W. to be Dem.*

PROP. XCIV.

See Scheme. 92.

If a space AC *be contained under a rationall line* AB *and a third residuall* AD (AE − DE) *the right line* TS *containing in power the space* AC *is a second mediall residuall line.*

As in the former, TO and SO are *μ.* Therefore because DE *a* is ρ' ⊓⊔ AB. *b* the rectangle DI, *c* and so DK, or TOS, shall be *μν. d* therefore TS = √ AC is a second mediall residuall. *W.W. to be Dem.* a hyp. b 21.10. c 24.10 d 76.10.

PROP. XCV.

See Scheme 92.

If a space AC *be conteined under a rationall line* AB *and a fourth residuall* AD (AE − DE) *the right line* TS *conteining the space* AC *in power, is a Minor line.*

As before, TO *a* ⊓⊔ SO. Therefore because AE *b* is ρ' ⊓⊔ AB, *c* shall AI (TOq + SOq) be ρ'ν. but, as before, the rectangle TOS is *μν. d* therefore TS = √ AC is a *Minor* line. *W.W. to be Dem.* a lem.91.10. b hyp. c 20 10. d 77.10.

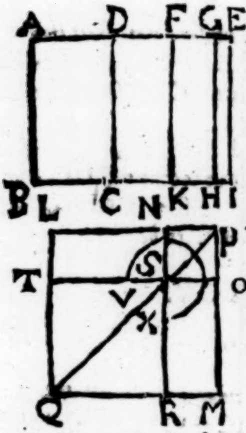
PROP. XCVI.

See Scheme 92.

If a space AC *be contained under a rationall line* AB *and a fift residuall* AD (AE − DE) *the right line* TS *containing in power the space* AC, *is a line which maketh with a rationall space the whole space mediall.*

For again TO ⊓⊔ SO. therefore since AE *a* is ρ' ⊓⊔ AB *b* also AI, that is TOq + SOq shall be *μν.* But, as in the 93. the rectangle TOS is ρ'ν. *c* whence TS = √ AC is a line which with ρ'ν makes a whole *μν.* *W.W. to be Dem.* a hyp. b 22.10. c 78.10.

PROP. XCVII.

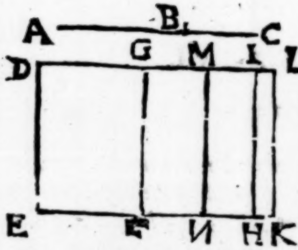


If a space AC *be contained under a rationall line* AB, *and a fixt residuall* AD (AE − DE) *the right line* TS *containing in power the space* AC, *is a line making with a mediall rectangle, a whole space mediall.*

As often above, TO ⊐ SO. also, as in 96, TOq + SOq is μν. but the rectangle TOS is ρ'ν, as in 94. a Lastly TOq + SOq ⊐ TOS. b therefore TS = √ AC is a line which with μν makes a whole μν. *W.W. to be Dem.*

a lem. 91. 10.
b 79. 10.

Lemma.



* cor. 16. 6.

Vpon a right line DE * *apply the rectangles* DF = ABq, *and* DH = ACq, *and* IK = BCq. *and let* GL *be bisected in* M, *and the line* MN *drawn parallel to* GF.

Then 1. *the rectangle* DK *is* = ACq + BCq. *as the construction manifests.*

2. *The rectangle* ACB = GN or MK. For DK a = ACq + BCq b = 2 ACB + ABq. but ABq a = DF. therefore GK c = 2 ACB. and d consequently GN or MK = ACB.

3. *The rectangle* DIL = MLq. For because ACq. ACB e :: ACB. BCq, that is DH. MK :: MK. IK.

a constr.
b 7. 2.
c 3. ax. 1.
d 7. ax. 1.
e 1. 6.

IK. *e* thence is DI. ML :: ML. IL. *f* therefore DIL f 17 6.
= MLq.

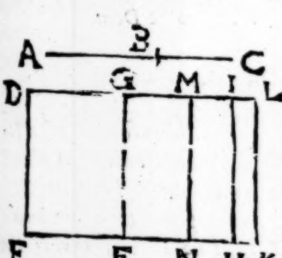
4. *If* AC *be taken* ⊐ BC, *then* DK *shall be* ⊓ ACq. For ACq + BCq (DK) *g* ⊓ ACq. g 16. 10.

5. *Likewise* DL ⊓ √ DLq − GLq. For because DH (ACq) ⊓ IK (BCq) *h* thence shall DI h 10. 10.
be ⊓ IL. *k* therefore √ DLq − GLq ⊓ k 18. 10.
DL.

6. *Also* DL ⊓ GL. For ACq + BCq ⊓ *l* 2 l *lem.* 26. 10.
ACB, that is, DK ⊓ GK. *m* therefore DL ⊓ m 10. 10.
GL.

7. *But if* AC *be taken* ⊐ BC, *n* *then* DL shall be n 19. 10
⊐ √ DLq − GLq.

Prop. XCVIII.



The square of a residuall line AB (AC − BC) *applied to rationall line* DE, *makes the breadth* DG *a first residuall line.*

Doe as is enjoined in the *Lemma* next preceding. Then because a *hyp.*
AC, BC, *a* are ρ′ ⊐. *b* also DK (ACq + BCq) shall b *lem.* 97. 10.
be ⊓ ACq. *c* therefore DK is ρν. *d* wherefore DL c *sch.* 12. 10
is ρ′ ⊓ DE *e* Likewise the rectangle GK (2 ACB) d 21. 10.
is μν. *f* therefore GL is ρ′ ⊓ DE. *g* and consequent- e 22, and 14 10.
ly DL ⊓ GL. *h* But DLq ⊓ GLq. *k* therefore f 23. 10.
DG is a residuall, *l* and that of the first order (be- g 13. 10.
cause *m* AC ⊐ BC, and therefore DL ⊓ √ h *sch.* 12. 10.
DLq − GLq.) *W.W. to be Dem.* k 74. 10. l 1. *def.* 85. 10. m *lem.* 97. 10.

PROP. XCIX.

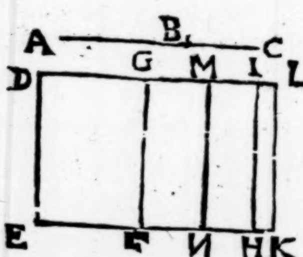
See the following Scheme.

The square of a first mediall residuall line AB (AC − BC) *applied to a rationall line* DE, *makes the breadth* DG *a second residuall line.*

a hyp. b lem.97.10. c 24.10. d 23.10. e hyp. & sch. 12.10. f 21.19. g 13.10. h sch.12.10. k 74.10. l lem 97 10. m 2.def. 85.10.

Supposing the foregoing *Lemma*; because AC and BC [a] are μ ⊐⊏, [b] thence shall DK (ACq + BCq) be ⊐⊏ ACq. [c] wherefore DK is μν. [d] therefore DL is ρ ⊐⊏ DE. [e] also GK (2 ACB) is ρν. [f] therefore GL is ρ ⊐⊏ DE; [g] wherefore DL ⊐⊏ GL. [h] But DLq ⊐⊏ GLq. [k] therefore DG is a residuall line; and because DL is ⊐⊏ √ DLq − GLq, [m] therefore shall DG be a second residuall. *W. W. to be Dem.*

PROP. C.



The square of a second mediall residuall line AB (AC − BC) *applied to a rationall line* DE, *makes the breadth* DG *a third residuall line.*

a 23.10 b lem.26.10. c 1.6 and 10.10. d sch.12.10. e 74.10. f 3.def.85.10. g lem.97.10.

Again DK is μν. [a] wherefore DL is ρ ⊐⊏ DE. also DGK is μν [a] whence GL is ρ ⊐⊏ DE. [b] likewise DK ⊐⊏ GK. [c] wherefore DL ⊐⊏ GL. [d] but DLq ⊐⊏ GLq. [e] therefore DG is a residuall line, and that of [f] the third order, [g] because DL ⊐⊏ √ DLq − GLq. *W. W. to be Dem.*

PROP. CI.

See the foregoing Scheme.

The square of a Minor line AB (AC − BC) *applied to*

to a rationall line DE , *makes the breadth* DG *a fourth residuall.*

As before, ACq + BCq, that is DK, is ῥν. a therefore DL is ῥ' ⌐⊔ DE. but the rectangle ACB, and so GK (2 ACB) * is μν. b wherefore GL is ῥ' ⌐⊔ DE. c therefore DL ⌐⊔ GL. d but DLq ⌐⊔ GLq. and because * ACq ⌐⊔ BCq, e thence shall DL be ⌐⊔ √ DLq − GLq. f therefore DG ha's the conditions required to a fourth residuall. *W.W. to be Dem.*

a 21.10.
* *hyp.*
b 23.10.
c 13.10.
d *sch.* 12. 10.
e *lem.* 97. 10.
f 4 *def.* 85. 10.

PROP. CII.

See Scheme 100.

The square of a line AB (AC − BC) *which makes with a rationall space the whole space mediall, applied to a rationall line* DE, *makes the breadth* DG *a fift residuall line.*

For, as above, DK is μν. a wherefore DL is ῥ' ⌐⊔ DE. also GK is ῥν. b whence GL is ῥ' ⌐⊔ DE. c therefore DL ⌐⊔ GL. d but DLq ⌐⊔ GLq. Moreover DL e ⌐⊔ √ DLq − GLq. wherefore DG f is a fift residuall. *W.W. to be Dem.*

a 23.10.
b 21.10.
c 13.10.
d *sch.* 13. 10.
e *lem.* 97. 10.
f 5. *def.* 85. 10.

PROP. CIII.

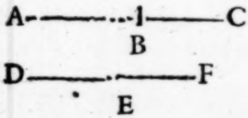
See the last Scheme.

The square of a line AB (AC − BC) *making with a mediall space the whole space mediall, applied to a rationall line* DE, *makes the breadth* DG *a sixt residuall line.*

As above DK and GK are μα; a wherefore DL and GL are ῥ' ⌐⊔ DE, also DK b ⌐⊔ GK. c whence DL ⌐⊔ GL. d therefore DG is a residuall. b And whereas ACq ⌐⊔ BCq. and so DL ⌐⊔ √ DLq − GLq, e therefore DG shall be a sixt residuall. *W.W. to be Dem.*

a 23.10.
b *hyp. and lem.* 97. 10.
c 10. 10.
d 74. 10.
e 6. *def.* 85. 10.

Prop. CIV.



A right line DE *commensurable in length to a residuall* AB (AC − BC) *is it self also a residuall, and of the same order.*

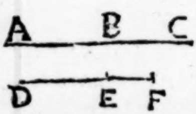
Lemma.

Let AB.DE :: AC.DF. *and* AB $\sqcap\!\sqcup$ DE.

I say AC + BC $\sqcap\!\sqcup$ DF + EF. For AC.BC [a] :: DF.EF. therefore by addition AC + BC.BC :: DF + EF.EF. therefore by inversion AC + BC.DF +
a *lem.*66.10. EF :: BC.EF. [a] but BC $\sqcap\!\sqcup$ EF. [b] therefore AC +
b 10.10. BC $\sqcap\!\sqcup$ DF + EF. *W.W. to be Dem.*

a 12. 6. [a] Make AB.DE :: AC.DF. [b] therefore AC + BC
b *lem.*103. 10. $\sqcap\!\sqcup$ DF + EF. therefore seeing AC + BC [c] is a bi-
c *hyp.* nomiall, [d] DF + EF shall be a binomiall too,
d 67. 10. and of the same order. [e] wherefore DF − EF is a re-
e *by def.*85. 10. siduall of the same order with AC − BC. *W.W. to be Dem.*

Prop. CV.

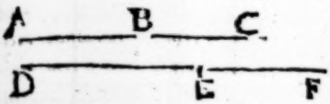


A right line DE *commensurable to a mediall residuall line* AB (AC − BC) *is it self a mediall residuall, and of the same ord r.*

a 12 6. Again [a] make AB.DE :: AC.DF. [b] whence AC
b *lem.*103. 10 + BC $\sqcap\!\sqcup$ DF + EF. [c] therefore DF + EF is a bi-
c 68.10. mediall of the same order with AC + BC, [d] and
d 75, *and* 76. 10. consequently DF − EF shall be a mediall residuall of the same order with AC − BC. *Which was to be Demonstrated.*

Prop.

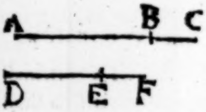
Prop. CVI.



A right line DE *commensurable to a Minor line* AB (AC – BC) *is it self also a Minor line.*

Make AB.DE :: AC. DF. [a] then is AC + BC $\sqcap\sqcup$ DF + EF. but AC + BC [b] is a *Major* line; [c] therefore DF + EF is also a *Major* line ; [d] and consequently DF – EF is a *Minor* line. *W. W. to be Dem.* a lem. 103. 10. b hyp. c 69. 10. d 77. 10.

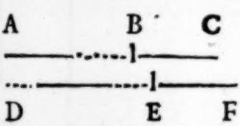
Prop. CVII.



A right line DE *commensurable to a line* AB (AC – BC) *which makes with a rationall space the whole space mediall, is it self also a line making with a rationall space the whole space mediall.*

For, accordingly as in the former, we may show DF + EF to contain in power ρν and μν. [a] whence DF – EF is a line making, &c. a 78. 10.

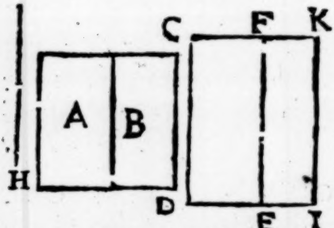
Prop. CVIII.



A right line DE *commensurable to a line* AB (AC – BC) *which with a mediall space makes the whole space mediall, is it self a line making with a mediall space the whole space mediall.*

For according to the preced. DF + EF shall contain in power 2 μα. [a] therefore DF – EF shall be, as in the *prop.* a 79. 10.

Prop. CIX.



A mediall rectangle B *being taken from a rationall rectangle* A+B, *the right line* H *which containeth in power the space remaining* A, *is one of these two irrational lines.* viz. *either a residuall line, or a Minor line.*

a 3. ax. 1.
b hyp. and constr.
c 21. 10.
d 23. 10.
e 13. 10.
f 74. 10.
g 1. def. 85. 10.
h 92. 10.
k 4 def. 85. 10.
l 95. 10.

Upon CD ρ' make the rectangles CI = A + B, and FI = B. whence CE *a* = A = Hq. wherefore because CI *b* is $\rho'\nu$. *c* therefore CK is ρ' ⊓⊔ CD. but being FI *b* is $\mu\nu$, *d* shall FK be ρ' ⊓⊔ CD. *e* whence CK ⊓⊔ FK. *f* therefore CF is a residuall line. Wherefore if CK be ⊓⊔ $\surd$ CKq − FKq, *g* then CF shall be a first residuall. *h* therefore $\surd$ CE (H) is a residuall line. But if CK ⊓⊔ $\surd$ CKq − FKq, *k* then CF shall be a fift residuall; and consequently H ($\surd$ CE) *l* shall be a *Minor* line. *W.W. to be Dem.*

Prop. CX.

See the prec. Scheme.

A rationall rectangle B *being taken away from a mediall rectangle* A + B, *other two irrationall lines are made, namely either a first mediall residuall line, or a line making with a rationall space the whole space mediall.*

a 3 ax. 1.
b hyp. and constr.
c 23 10.
d 21. 10.
e 13. 10.
f 74 10
g 2. def. 85. 10.
h 93 10.
k 5 def. 85. 10.
l 96. 10.

Upon CD the propounded ρ' make the rectangles CI = A + B, and FI = B. *a* whence CE = A = Hq. Therefore because CI *b* is $\mu\nu$; *c* shall CK be ρ' ⊓⊔ CD. but because FI *b* is $\rho'\nu$. *d* thence FK ρ' ⊓⊔ CD. *e* whence CK ⊓⊔ FK. *f* therefore CF is a residuall, *g* and that a second. If CK ⊓⊔ $\surd$ CKq − FKq, *h* then H ($\surd$ CE) is a first mediall residuall. But if CK ⊓⊔ + CKq − FKq, *k* then shall CF be a fift residuall; and *l* consequently H ($\surd$ CE) shall be a line making $\mu\nu$ with $\rho'\nu$. *W.W. to be Dem.*

Prop.

PROP. CXI.

See the same Scheme.

A mediall space B *being taken away from a mediall space* A + B , *which is incommensurable to the whole* A + B, *the other two irrationall lines are made; viz. either a second mediall residuall line, or a line making with a mediall space the whole space mediall.*

Upon CD ρ' make the rectangles CI = A + B, and FI = B. a wherefore CE = A = Hq. Because a 3.ax.1.
therefore CI is μν, b thence CK is ρ' ˥⊓⊥ CD. and in b 23.10. c hyp.
like manner FK ρ' ⊓⊥ CD. Likewise because CI d 10.10.
c ˥⊓⊥ FI, d therefore CK ˥⊓⊥ FK. e wherefore CF is e 74.10.
a residuall, f namely a third. If CK ˥⊓⊥ √ CKq − f 3.def 85.10
FKq, g whence H (√ CE) shall be a second mediall g 94.10.
residuall. but if CK ˥⊓⊥ √ CKq − FKq h then shall h 6.def.85.10.
CF be a sixt residuall. k wherefore A shall be a line k 97.10.
making μν with μν. *W.W. to be Dem.*

PROP. CXII.

A residuall line A *is not the same with a binomiall line.*

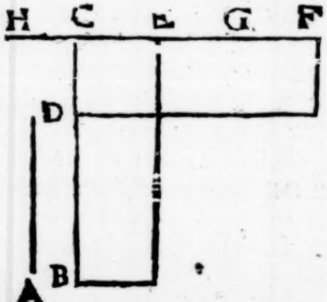
Upon BC propounded ρ' make the rectangle CD = Aq. Therefore seeing A is a residuall, a BD shall a 98.10.
be a first residuall, to which let DE be the line congruent or that may be adjoined. b wherefore BE, DE, b 74.10.
are ρ' ⊓⊥. c and BE ⊓⊥ BC. If you conceive A to c 1.def.85.10.
be a binomiall, then BD is a first bin. whose names
let be BF, FD; and let BF be ⊏ FD. d therefore BF, d 37.10.
FD are ρ' ⊓⊥; and BF e ⊓⊥ BC. therefore since BC e 1.def.48.10.
⊓⊥ BE, f shall BE be ⊓⊥ BF. g and thence BE ⊓⊥ f 12.10.
FE. h therefore FE is ρ'. Likewise because BE ⊓⊥ g cor. 16.10.
DE, k shall FE be ⊓⊥ DE. l wherefore FD is a resi- h sch 12.10. k 14.10.
duall, & so FD is ρ'. but it was shewn ρ'. which are re- l 74.10.
pugnant. Therefore A is falsely conceived to be a binomiall. *W.W. to be Dem.*

The names of the 13. irrationall lines differing one from another.

1. A Mediall line.
2. A binomial line; of which there are six species.
3. A first bimediall line.
4. A second bimediall.
5. A Major line.
6. A line containing in power a rationall superficies and a mediall superficies.
7. A line containing in power two mediall superficies.
8. A residuall line; of which there also six kinds.
9. A first mediall residuall line.
10. A second mediall residuall line.
11. A Minor line.
12. A line making with a rationall superficies the whole superficies mediall.
13. A line making with a mediall superficies the whole superficies mediall.

Being the differences of breadths do argue differences of right lines, whose squares are applied to some rationall line, and it is demonstrated in the preced. Propositions that the breadths which arise from applying of the squares of these 13. lines do differ one from another, it evidently follows that these 13. lines do also differ one from another.

PROP. CXIII.



The square of a rationall line A *applied to a binomiall* BC (BD + DC) *makes the breadth* EC *a residuall line, whose names* EH, CH, *are commensurable to the names* BD, DO, *of the binomiall line, and in*

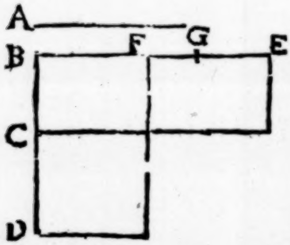
in the same proportion (EH. BD :: CH.DC:) *and moreover, the residuall line* EC *which is made, is of the same order with* BC *the binomiall.*

Upon DC the lesse name a make the rectangle DF = Aq = BE. whence BC.CD b :: FC. CE. therefore by division, BD DC :: FE. EC. And whereas BD c ⊏ DC, d thence FE shall be ⊏ EC. Take EG = EC, & make FG.GE :: EC.CH. Then EH, & CH shall be the names of the residuall EC, whereunto all is agreeable that is propounded in the theoreme. For being that by addition FE. GE (EC) :: EH. CH. therefore FH. EH e :: EH. CH f :: FE. EC f :: BD. DC. wherefore since BD g ⊐⊏ DC. h thence shall EH be ⊐⊏ CH, i and FHq ⊐⊏ EHq. Therefore because FHq. EHq k :: FH. CH. l shall FH be ⊐⊏ CH, l and so FC ⊐⊏ CH. Moreover CD g is ρ', and DF (Aq) g is ρ'v. m therefore FC is ρ', ⊐⊏ CD. whence also CH is ρ' ⊐⊏ CD. n therefore EH, CH are ρ' and ⊐⊏, as before. o therefore EC is a residuall line, to which CH may be joined. Furthermore EH.CH f :: BD.DC. and so by inversion EH.BD :: CH.DC. whence because CH f ⊐⊏ DC, p shall EH be ⊐⊏ BD. But suppose BD ⊐⊏ √ BDq − DCq. q then shall EH be ⊐⊏ √ EHq − CHq. Also if BD ⊐⊏ ρ' propounded, then shall EH be ⊐⊏ to the same ρ'. f that is, if BC be a first binomiall, r EC shall be a first residuall. In like manner, if DC be to the ⊐⊏ propounded ρ', r then is CH ⊐⊏ to the same ρ'. u that is, if BC be a second binomial, x EC shall be a second residuall: and if this be a third binom. then that shall be a third residuall, &c. But if BD be ⊐⊏ √ BDq − DCq, y then shall EH be ⊐⊏ √ EHq − CHq. therefore if BC be a 4, 5, or 6 binomiall, EG shall be likewise a 4, 5, or 6 residuall.

W.W. to be Dem.

a cor. 16 6.
b 14 6.
c hyp.
d 14 5.
e 12. 5.
f before.
g hyp.
h 10. 10.
k cor. 20. 6.
l 16 10.
m 21 10.
n sch. 12. 10.
o 74. 10.
p 10. 10.
q 15. 10.
r 12. 10.
s 1. def. 48. 10.
t 1. def. 85. 10.
u 2. def. 48. 10.
x 2. def. 85. 10.
y 15. 10.

PROP. CXIV.

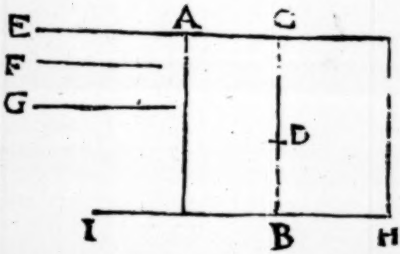


The square of a rationall line A *applied to a residuall line* BC (BD — CD) *makes the breadth* BE *a binomiall; whose names* BE, GE *are commensurable to the names* BD, BC *of the residuall line* BC, *& in the same proportion. and moreover, the binomiall line which is made* (BE) *is of the same order with the residuall line* (BC.)

a cor. 16.6. *a* Make the rectangle DF = Aq. and BF.FE *b* ::
b 12. 6. EG.GF. whence for that DF = Aq = CE, *c* there-
c 14. 6. fore BD. BC :: BE. BF. therefore by conversion of
d 19. 5. proportion BD. CD :: BE. FE :: EG. GF :: *d* BG.
e hyp. EG. but BD *e* ⊤ CD. *f* therefore BG ⊤ GE.
f 10. 10. therefore because BGq. GEq *g* :: BG.GF. *h* shall BG
g cor. 20.6. be ⊓ GF. *k* and so BG ⊓ BF. moreover BD *e* is
h 10.10. ρ', and the rectangle DF(Aq) *e* is ρ'v. *l* therefore BF is
k cor. 16.10. ρ' ⊓ BD. *m* therefore also BG is ρ' ⊓ BD. *n* therefore
l 21. 10. BG, GE are ρ' ⊤. *o* wherefore BE is a binomiall.
m 12.10. Lastly, because BD.CD :: BG. GE. and inversely
n sch. 12. 10. BD.BG :: CD. GE. and BD ⊓ BG. *p* thence shall
o 37. 10. CD be ⊓ GE. therefore if CB be a first residuall,
p 10. 10. BE shall be a first binomiall, &c. as in the *prec.*
therefore, &c.

PROP.

PROP. CXV.



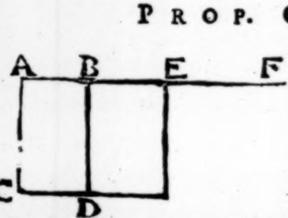
If a space AB *be contained under a residuall line* AC (CE − AE) *and a binomiall* CB, *whose names* CD, DB *are commensurable to the names* CE, AE, *of the residuall line, and in the same proportion* (CE. AE :: CD. DB) *then the right line* F *which containeth in power that space* AB, *is irrationall.*

Let G be ῥ. and make the rectangle CH = Gq; *a* then shall BH (HI − IB) be a residuall line, and
HI *a* ⊐⊏ CD *b* ⊐⊏ CE. *a* and BI ⊐⊏ DB. *a* and HI. a 113. 10.
BI :: CD. DB *b* :: CE. EA. therefore by inversion b *hyp.*
HI. CE :: BI. EA. *c* therefore BH. AC :: HI. c 19. 5.
CE :: BI. EA. wherefore since *d* HI ⊐⊏ CE, d 12. 10.
e thence BH ⊐⊏ AC. *f* therefore the rectangle HC ⊐⊏ e 10. 10.
BA. But HC (Gq) *b* is ῥν. *g* therefore BA (Fq) is ῥν: f 1. 6 *and*
and consequently F is ῥ. *W.W. to be Dem.* g *sch* 12. 10.

Coroll.

Hereby it appears that a rationall superficies may be contained under two irrationall right lines.

PROP. CXVI.



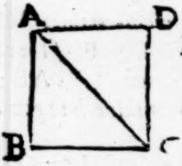
Of a mediall line AB *are produced infinite irrationall lines* BE, EF, &c. *whereof none is of the same kind with any of the precedent.*

Let AC be propounded ῥ. & AD a rectangle contained

tained under AC, AB. [a] therefore AD is $\rho\nu$. Take BE $= \sqrt{}$ AD. [b] then BE is ρ', and the same with none of the former. For no square of any of the former being applied to ρ', makes the breadth mediall. Let the rectangle DE be finished, [a] then DE shall be $\rho\nu$, and [b] consequently EF ($\sqrt{}$ DE) shall be ρ', and not the same with any of the former. for no square of the former being applyed to ρ', makes the latitude BE. therefore, &c.

a lem 38. 10
b 11. 10.

Prop. CXVII.



Let it be required to shew that in square figures BD, *the diameter* AC *is incommensurable in length to the side* AB.

For ACq. ABq [a] :: 2. 1 [b] :: not Q. Q. [c] therefore AC ⊓⊔ AB. *W. W. to be Dem.* This theoreme was of great note with the ancient philosophers; so that he that understood it not was esteemed by *Plato* undeserving the name of a man, but rather to be reckoned among brutes.

a 47. 1.
b cor. 24. 8.
c 9. 10.

The End of the tenth Book.

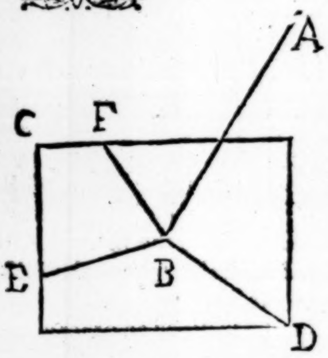
THE

THE ELEVENTH BOOK OF EUCLIDE'S ELEMENTS.

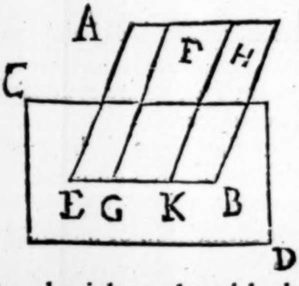
Definitions.

I. A Solid is that which hath length, breadth, and thicknesse.

II. The terme, or extreme, of a solid is a Superficies.

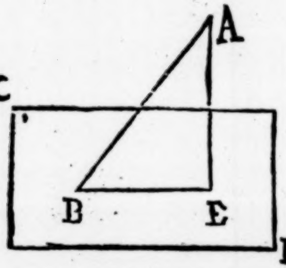


III. A right line AB is perpendicular to a Plane CD, when it makes right angles ABD, ABE, ABF with all the right lines BD, BE, BF, that touch it, and are drawn in the said Plane.

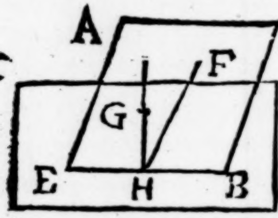


IV. A Plane AB, is perpendicular to a Plane CD, when the right lines FG, HK, drawn in one Plane AB to the line of common section of the two Planes EB, and making right angles therewith, do also make right angles with the other Plane CD.

V. The



V. The inclination of a right line AB to a Plane CD, is, when a perpendicular AE is drawn from A the highest point of that line AB to the plane CD, and another line EB drawn from the point E, which the perpendicular AE makes in the Plane CD, to the end B of the said line AB which is in the same Plane, whereby the angle is acute ABE which is contained under the insisting line AB, and the line drawn in the plane EB.



VI. The inclination of a Plane AB to a Plane CD, is an acute angle FHG contained under the right lines FH, GH, which being drawn in either of the Planes AB, CD to the same point H of the common section BE, make right angles FHB, GHB, with the common section BE.

VII. Planes are said to be inclined to other planes in the same manner, when the said angles of inclination are equall one to another.

VIII. Parallel Planes are those which being prolonged never meet.

IX. Like solid figures are such as are contained under like planes equall in number.

X. Equall and like solid figures are such as are conteined under like planes equall both in multitude and magnitude.

XI. A solid angle is the inclination of more then two right lines which touch one another, and are not in the same superficies.

Or thus;

A ſolid angle is that which is contained under more then two plane angles not being in the ſame ſuperficies, but conſiſting all at one point.

XII. A Pyramide is a ſolid figure comprehended under divers planes ſet upon one plane, (which is the baſe of the pyramide,) and gathered together to one point.

XIII. A Priſme is a ſolid figure contained under planes, whereof the two oppoſite are equall, like, and parallel; but the others are parallelograms.

XIV. A Sphere is a ſolid figure made when the diameter of a ſemicircle abiding unmoved, the ſemicircle is turned round about, till it return to the ſame place from whence it began to be moved.

Coroll.

Hence, all the rayes drawn from the center to the ſuperficies of a ſphere, are equall amongſt themſelves.

XV. The Axis of a ſphere, is that fixed right line, about which the ſemicircle is moved.

XVI. The Centre of a Sphere, is the ſame point with that of the ſemicircle.

XVII. The Diameter of a Sphere, is a right line drawn though the centre, and terminated on either ſide in the ſuperficies of the ſphere.

XVIII. A Cone is a figure made, when one ſide of a rectangled triangle (*viz.* one of thoſe that contain the right angle) remaining fixed, the triangle is turned round about till it return to the place from whence it firſt moved. And if the fixed right line be equall to the other which containeth the right angle, then the Cone is a rectangled cone: but if it be leſſe, it is an obtuſe-angled Cone; if greater, an acute-angled Cone.

XIX. The Axis of a Cone is that fix'd line about which the triangle is moved.

XX. The

XX. The Base of a Cone is the circle, which is described by the right line moved about.

XXI. A Cylinder is a figure made by the moving round of a right-angled parallelogram, one of the sides thereof, (namely which contain the right angle) abiding fix'd, till the parallelogram be turned about to the same place, where it began to move.

XXII. The Axis of a Cylinder is that quiescent right line, about which the parallelogram is turned.

XXIII. And the Bases of a Cylinder are the circles which are described by the two opposite sides in their motion.

XXIV. Like Cones and Cylinders, are they, both whose Axes and Diameters of their Bases are proportionall.

XXV. A Cube is a solid figure contained under six equall squares.

XXVI. A Tetraedron is a solid figure contained under four equall and equilaterall triangles.

XXVII. An Octaedron is a solid figure contained under eight equall and equilaterall triangles.

XXVIII. A Dodecaedron is a solid figure contained under twelve equall, equilaterall and equiangular Pentagones.

XXIX. An Icosaedron is a solid figure contained under twenty equall and equilaterall triangles.

XXX. A Parallelipedon is a solid figure contained under six quadrilaterall figures, whereof those which are opposite are parallell.

XXXI. A solid figure is said to be inscribed in a solid figure, when all the angles of the figure inscribed are comprehended either within the angles, or in the sides, or in the planes of the figure wherein it is inscribed.

XXXII. Likewise a solid figure is then said to be circumscribed about a solid figure, when either the angles, or sides, or planes of the circumscribed figure touch all the angles of the figure which it contains.

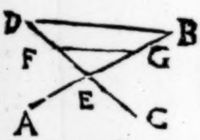
PRO-

PROPOSITION I.

One part AC *of a right line cannot be in a plane superficies, and another part* CB *elevated upward.*

Produce AC in the plane directly to F. If you conceive CB to be drawn strat from AC, then two right lines AB, AF, have one common segment AC. *a Which is impossible.* a 10 *ax.* 1.

PROP. II.



If two right lines AB, CD, *cut one another, they are in the same plane: And every Triangle* DEB *is in one and the same plane.*

For imagine EFG, part of the triangle DEB, to be in one plane, and the part FDGB to be in another. then EF part of the right line ED is in a plane, and the other part elevated upwards. *a Which is Absurd.* Therefore the triangle EDB is in one and the same plane; and so also are the right lines ED, EB; *a* wherefore the whole lines AB, DC, are in one plane. *Which was to be Demonstrated.* a 1. 11.

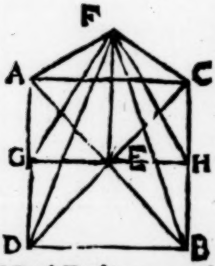
PROP. III.

If two planes AB, CD, *cut one the other, their common section* EF *is a right line.*

If EF the common section be not a right line, *a* then in the plane AB draw the right line EGF, *a* and in the plane CD the right line EHF. therefore two right lines EGF, EHF include a superficies. *b Which is Absurd.* a 1 *post* 1. b 14 *ax.* 1.

PROP.

PROP. IV.

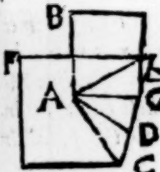


If a right line EF *be at right angles erected upon two lines* AB, CD, *cutting one the other, at the common section* E; *it shall also be at right angles to the plane* ACBD *drawn by the said lines.*

Take EA, EC, EB, ED, equall one to the other, and join the right lines AC, CB, BD, AD. draw any right line GH through E, & join
a constr. FA, FC, FD, FB, FG, FH. Because AE is *a* = EB,
b 15. 1. and DE *a* = EC, and the angle AED *b* = CEB,
c 4. 1. *c* therefore AD is = CB, *c* & likewise AC = DB.
d sch. 34. 1. *d* therefore AD is parallel to CB, *d* and AC to DB.
e 29. 1. *e* wherefore the angle GAE = EBH, and the angle
f constr. AGE = EHB. But also AE *f* = EB. *g* therefore GE
g 26. 1. = EH, *g* and AG = BH. whence by reason of the
h 4. 1. right angles, by the hyp. and so equall, at E, *h* the bases FA, FC, FB, FD, are equall. Therefore the triangles ADF, FBC, are equilaterall one to another. *k* and thence the angle DAF = BCF. There-
k 8. 1. fore in the triangles AGF, FBH, the sides FG, FH
l 4. 1. *l* are equall; and so by consequence the triangles
m 8. 1. FEG and FEH are mutually equilaterall. *m* there-
n 10. def. 1. fore the angles FEG, FEH are equall, and *n* so right angles. In like manner, FE makes right angles with all the lines drawn through E in the plane ADBC,
o 3. def. 11. *o* and is therefore perpendicular to the said plane.

PROP.

PROP. V.

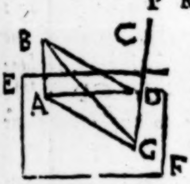


If a right line AB *be erected perpendicular to three right lines* AC, AD, *touching one the other at the common section, those three lines are in the same plane.*

For AC, AD, *a* are in one plane FC; *a* and AD, AE, are in one plane BE. which if you conceive to be severall planes, then let their intersection *b* be the right line AG; therefore because BA by the Hypoth. is perpendicular to the right lines AC, AD. *c* and so to the plane FC, *d* it is also perpendicular to the right line AG. therefore (since that AB is in the same plane with AC, AE) the angles BAG, BAE, are right angles, and consequently equall, the part & the whole. *Which is Absurd.*

a 2. 11.
b 3. 11.
c 4. 11.
d 3. def. 11.

PROP. VI.

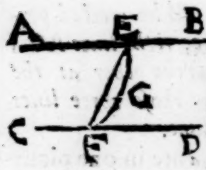


If two right lines AB, DC, *be erected perpendicular to one and the same plane* EF, *those right lines* AB, DC *are parallel one to the other.*

Draw AD, whereunto let DG = AB be perpendicular in the plane EF, and join BD, BG, AG. Being in the triangles BAD, ADG, the angles DAB, ADG *a* are right angles, and AB *b* = DG, and AD is common, *c* therefore BD is = AG. whence in the triangles AGB, BGD, equilaterall one to the other, the angle BAG is *d* = BDG; of which being BAG is a right angle, BDG shall be so also. but the angle GDC is supposed right, therefore the right line GD is perpendicular to the three lines DA, DB, CD. *e* which are therefore in the same plane *f* wherein AB is. Wherefore since AB and CD are in the same plane, and the internall angles BAD, CDA, are right angles, *g* AB and CD shall be parallels. *W.W. to be Dem.*

a hyp.
b constr.
c 4. 1.
d 8. 1.
e 5. 11.
f 2. 11.
g 28. 1.

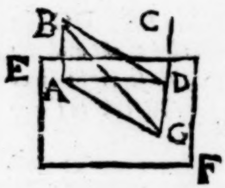
PROP. VII.



If there be two parallel right lines AB, CD, *and any points* E, F, *be taken in both of them, the line* EF *which is joined at these points, is in the same plane with the parallels* ABCD.

Let the plane in which AB, CD, are, be cut by another plane at the points E, F. then if EF is not in the plane ABCD, it shall not be the common section. Therefore let EGF be the common section; a 3.11. which *a* then is a right line. therefore two right b 14.ax.1. lines EF, EGF, include a superficies. *b Which is Absurd.*

PROP. VIII.

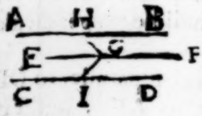


If there be two parallel right lines AB, CD, *whereof one* AB *is perpendicular to a plane* EF. *then the other* CD *shall be perpendicular to the same plane* EF.

The preparation and demonstration of the sixt of this Book being transferr'd hither; the angles GDA, and a 4. 11. GDB are right angles: *a* therefore GD is perpendi- b 7. 11. cular to the plane, wherein are AD, DB (*b* in which c 3 def. 11. also AB, CD, are.) *c* therefore GD is perpendicular to CD. but the angle CDA is also *d* a right an- d 29. 1. e 4. 11. gle. *e* therefore CD is perpendicular to the plane EF. *W. W. to be Dem.*

PROP.

PROP. IX.



Right lines (AB,CD) which are parallel to the same right line EF, but not in the same plane with it, are also parallel one to the other.

In the plane of the parallels AB, EF, draw HG perpendicular to EF; also in the plane of the parallels EF,CD, draw IG perpendicular to EF. *a* therefore EG is perpendicular to the plane wherein HG, GI are; and AH, CI are perpendicular to the same plane. *c* therefore AH and CI are parallels. *W.W.to be Dem.* a 4. 11. b 8. 11. c 6. 11.

PROP. X.

If two right lines AB, AC, touching one another be parallel to two other right lines ED, DF, touching one another, and not being in the same plane, those right lines contain equall angles, BAC, EDF.

Let AB, AC, DE, DF, be equall one to the other, and draw AD, BC, EF, BE, CF. Being AB, DE, *a* are parallels and equall, *b* also BE, AD, are parallels and equall. In like manner CF, AD, are parallels and equall; *c* therefore also BE, FC, are parallels and equall. *d* Therefore BC, EF are equall. Wherefore since the triangles BAC, EDF, are of equall sides one to the other, the angles BAC, EDF *e* shall be equall. *W.W.to be Dem.* a hyp. and constr. b 33. 1. c 2. ax. 1. & 30. 1. d 33. 1. e 8. 1.

PROP. XI.

From a point given on high A to draw a right line AI perpendicular to a plane below BC.

In the plane BC draw any

 line

a 12.1. b 11.1. line DE; to which from the point A *a* draw the perpendicular AF, and *b* likewise FH in the plane BC cutting the said line DE at F; *a* then let fall AI perpendicular to FH. Which AI shall be perpendicular to the plane BC.

c 31.1. d constr. e 4.11. f 8.11. g 3.def.11 h constr. l 4.11. For through I *c* let KIL be drawn parallel to DE. Because DE *d* is perpendicular to AF, and FH, *e* therefore DE shall be perpendicular to the plane IFA. and so also KL *f* is perpendicular to the same plane. *g* therefore the angle KIA is a right angle. but the angle AIF is also *h* a right angle. *l* therefore AI is perpendicular to the plane BC. *W.W. to be Done.*

Prop. XII.

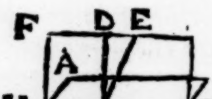
In a plane given BC, *at a point given therein* A, *to erect a perpendicular line* AF.

a 11.11. From some point without the plane, D, *a* draw DE perpendicular to the said plane BC. and joining the points A, E, by a line AE, *b* draw AF parallel to DE. *c* it is apparent that AF is perpendicular to the plane BC. *W.W. to be Done.*

b 31.1. c 8.11.

This and the preceding probleme are practically performed by applying two Squires to the point given; as appears by 4.11.

Prop. XIII.



At a point given C *in a plane given* AB, *two right lines* CD, CE, *cannot be erected perpendicular on the same side.*

a 6.11. For both CD, and CE, *a* should then be perpendicular to the plane AB, and consequently parallels; which is repugnant to the definition of parallel lines.

Prop.

PROP. XIV.

Planes CD , FE , *to which the same right line* AB *is perpendicular, are parallel.*

If you deny this ; then let the planes CD, FE, meet , so that their common section be the right line GH. in which take any point I, draw to it the right lines IA, IB, in the said planes. whereby in the triangle IAB , two angles IAB , IBA a are right angles. b *Which is Absurd.* a hyp. and 3. def 11. b 17. 1.

PROP. XV.

If two right lines AB, AC, *touching one the other, be parallel to two other right lines* DE, DF, *touching one the other , and not being in the same plane with them , the planes* BAC , EDF, *drawn by those right lines are parallel one to the other.*

From A a draw AG perpendicular to the plane EF. b and let GH, GI be parallel to DE, DF. c these also shall be parallel to AB, AC. Therefore since the angles IGA , HGA , d are right angles, also CAG, BAG, e shall be right angles. f therefore GA is perpendicular to the plane BC ; but the same is perpendicular to the plane EF. h therefore the planes BC, EF, are parallel. *W. W. to be Dem.*

a 11. 11.
b 31. 1.
c 30. 1.
d 3. def. 11.
e 29. 1.
f 4. 11.
g constr.
h 14. 11.

PROP. XVI.

If two parallel planes AB, CD, *be cut by some other plane* HEIGF, *their common sections* EH, GF *are parallel one to the other.*

For if they be conceived to be otherwise; being in the same plane that cuts them, they will meet some where, if produced; suppose in I; wherefore since the whole lines HEI, FGI a are in the planes AD, CD, being produced, the planes also shall meet. *contrary to the Hyp.*

a 1. 11.

PROP. XVII.

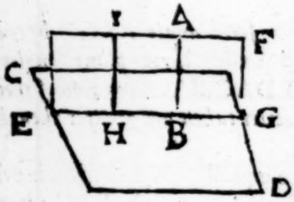
If two right lines ALB, CMD, *be cut by parallel planes* EF, GH, IK; *they shall be cut proportionally,* (AL. LB :: CM. MD.)

Let the right lines AC, BD, be drawn in the planes EF, IK; as also AD passing through the plane GH in the point N. and join NL, LM. the planes of the triangles ADC, ADB, make the sections BD, LN, and AC, NM a parallels. Therefore AL. LB :: AN. ND b :: CM. MD. *W.W. to be Dem.*

a 16. 11.
b 2. 6.

PROP.

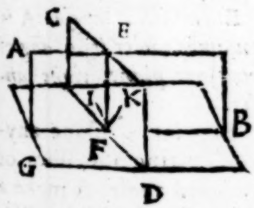
Prop. XVIII.



If a right line AB *be perpendicular to some plane* CD, *all the planes extended by that right line* AB (EF, &c.) *shall be perpendicular to the same plane* CD.

Let there be some plane EF drawn by AB, making the section EG with the plane CD; from some point whereof H, *a* draw HI parallel to AB in the plain EF; *b* then shall HI be perpendicular to the plane CD, and so likewise any other lines, that are perpendicular to EG. a 31. 1.
c therefore the plane EF is perpendicular to the b 8. 11.
plane CD; and by the same reason any other planes c 4. *def.* 11.
drawn by AB shall be perpendicular to EF. *W. W. to be Dem.*

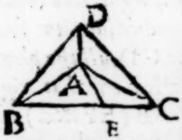
Prop. XIX.



If two planes AB, CD, *cutting one the other, be perpendicular to some plane* GH, *their line of common section* EF *shall be perpendicular to the same plane* (GH.)

Because the planes AB, CD, are taken perpendicular to the plane GH, it appears by 4. def. 11. that out of the point F there may be drawn in both planes AB, CD, a perpendicular to the plane GH. which shall be *a* but one; and a 13. 11.
therefore the common section of the said planes. *W. W. to be Dem.*

PROP. XX.



If a solid angle ABCD *be contained under three plane angles,* BAD, DAC, BAC, *any two of them howsoever taken are greater then the third.*

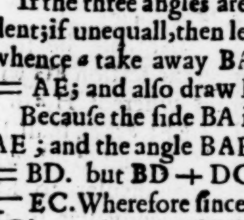
If the three angles are equall, the assertion is evident; if unequall, then let the greatest be BAC; from whence *a* take away BAE = BAD, and make AD = AE; and also draw BEC, BD, DC.

a 23. 1.

Because the side BA is common, and AD *b* = AE; and the angle BAE *b* = BAD, *c* thence is BE = BD. but BD + DC is *d* $\sqsubset$ BC. *e* therefore DC $\sqsubset$ EC. Wherefore since AD *b* = AE, and the side AC is common, and DC $\sqsubset$ EC. *f* the angle CAD shall be $\sqsubset$ EAC. *g* therefore the angle BAD + CAD $\sqsubset$ BAC. *W.W. to be Dem.*

b constr.
c 4 1.
d 20. 1.
e 5. ax. 1.
f 25. 1.
g 4. ax. 1.

PROP. XXI.



Every solid angle A *is contained under lesse angles then four plane right angles.*

For let a plane anywise cutting the sides of the solid angle A make a many-sided figure BCDE, and as many triangles ABC, ACD, ADE, AEB. I denote all the angles of the polygone by X; and I term the summe of the angles at the bases of the triangles Y. wherefore X + 4 Right ang. *a* = Y + A. but being that, (of the angles at B) *b* the angle ABE + ABC is $\sqsubset$ CBE, and the same is true also of the angles at C, at D, and at E, *c* it is manifest that Y is $\sqsubset$ X. and

a 32. 1. and sch. 32. 1.
b 20. 11.
c 5. ax 1.

and consequently A shall be $\sqsupset$ 4 Right ang. *W.W. to be Dem.*

PROP. XXII.

If there be three plane angles A , B , HCI , *whereof two howsoever taken are greater then the third , and the right lines which contain them be equall* AD, AE, FB, &c. *then of the right lines* DE, FG, HI , *coupling those equall right lines together , it is possible to make a triangle.*

A triangle may be [a] made of them , if any two be greater then the third: but they are so. For [b] make the angle HCK $=$ B , and CK $=$ CH , and draw HK, IK. [c] thence KH $=$ FG. and because the angle KCI [d] $\sqsubset$ A. [e] therefore KI $\sqsubset$ DE. but KI [f] $\sqsupset$ HI $+$ KH (FG.) therefore DE $\sqsupset$ HI $+$ FG. By the like argument any two may be proved greater then the third ; and consequently [a] it is possible to make a triangle of them. *W.W. to be Dem.*

a 22. 1.
b 23. 1.
c 4. 1.
d hyp.
e 24. 1.
f 20. 1.

PROP.

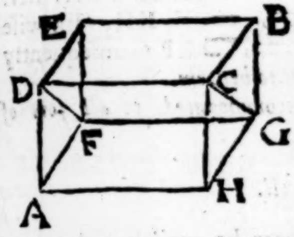
PROP. XXIII.

To make a solid angle MHIK *of three plane angles* A, B, C, *whereof two howsoever taken are greater then*
*21. 11. *the third.* * *But it is necessary that those three angles be lesse then four right angles.*

Make AD, AE, BE, BF, CF, CG, equall one to the other; and of the subtended lines DE, EF, FG (that
a 22. 1. and 21. 1. is, of the equall lines HI, IK, KH) *a* make the triangle HKI; about which *b* describe the circle LHKI.
b 5. 4.
* See Clavius. * But because AD is $\sqsubset$ HL. *c* let ADq be $=$ HLq $+$
c sch. 47. 1. LMq. *d* and let LM be perpendicular to the plane of
d 12. 11. the circle HKI. and draw HM, KM, IM. wherefore
e 3. def. 11. since the angle HLM *e* is a right angle, *f* thence is
f 47. 1. MHq $=$ HLq $+$ LMq *g* $=$ ADq. therefore MH $=$
g constr. AD. By the same reason MK, MI, AD (that is AE, EB, &c.) are equall; therefore since HM $=$ AD, and
h constr. MI $=$ AE, and DE *h* $=$ HI, *k* the angle A shall be
k 8. 1. $=$ HMI, *k* as likewise the angle IMK $=$ B, *k* and the angle HMK $=$ C. wherefore a solid angle is made at M of the three given plane angles. *W.W. to be Done.* AD is assumed to be $\sqsubset$ HL. But this is manifest. For if AD be $=$ or $\sqsupset$ HL, then is the angle A
l constr. and 8. 1. *l* $=$ *m* or $\sqsubset$ HLI. In like manner shall B be equall
m 21. 1. or $\sqsubset$ HLK, and C $=$ or $\sqsubset$ KLI. wherefore A $+$ B
*4 cor. 13. 1. $+$ C * shall either equall or exceed four right angles. *contrary to the Hypoth.* therefore rather let AD be $\sqsubset$ HL, *W.W. to be Dem.*

PROP.

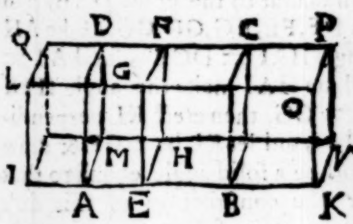
PROP. XXIV.



If a solid AB be contained under parallel planes, the opposite planes thereof (AG, DB, &c.) are like and equall parallelograms.

The plane AC cutting the parallel planes AG, DB, a makes the sections AH, DC, parallels. and by the same reason AD, HC are parallels. Therefore ADCH is a pgr. By the like argument the other planes of the parallelepipedon are b pgrs. wherefore being AF is parallel to HG, and AD to HC, c the angle FAD shall be = CGH. therefore because AF d = HG, and AD d = HC, and so AF. AD :: HG. HC. the triangles FAD, GHC, g are like and h equall; and consequently the pgrs. AE, HB are like and k equall. and the same may be shewn of the rest opposite planes. therefore, &c.

a 16. 11.
b 35. def. 1.
c 10. 11.
d 34. 1.
e 7. 5.
g 6. 6.
h 4. 1.
k 6. ax. 1.

PROP. XXV.

If a solid Parallelepipedon ABCD be cut by a plane EF parallel to the opposite planes AD, BC; then as the base AH is to the base BH, so shall solid AHD be to solid BHC.

Conceive the Parallelepipedon to be extended on either side. and take AI = AE, and BK = EB, and put the planes IQ, KP, parallel to the planes AD, BC; then the pgrs. IM, AH, and a DL, DG, b and IQ, AD, EF, &c. are a like and equall. c wherefore the Parallelepipedon AQ is = AF; and by the same reason

a 36. 1. and 1 def. 6.
b 24. 11.
c 10. def. 11.

reaſon the Parallelepipedon BP = BF. therefore the ſolids IF, EP are as multiplex of the ſolids AF, EC, as the baſes IH, KH, are of the baſes AH, BH.

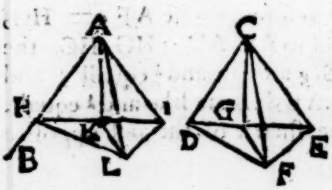
d 24.11. and 9 def. 11. e 6. def. 5. And if the baſis IH be $\sqsubset$, =, $\sqsupset$ KH, d likewiſe ſhall the ſolid IE be $\sqsubset$, =, $\sqsupset$ EP. e conſequently AH.BH :: AF.EC. *W.W. to be Dem.*

The ſame may be accommodated to all ſort of priſmes, whence

Coroll.

If any priſme whatſoever be cut by a plane parallel to the oppoſite planes, the ſection ſhall be a figure equall and like to the oppoſite planes.

Prop. XXVI.

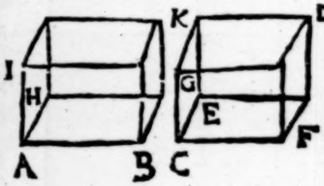


Upon a right line given AB, *and at a point given in it* A, *to make a ſolid angle* AHIL *equall to a ſolid angle given* CDEF.

a 11. 11. From ſome point F a draw FG perpendicular to the plane DCE, and draw the right lines DF, FE, EG, GD, CG. Make AH = CD, and the angle HAI = DCE, and AI = CE; and in the plane HAI make the angle HAK = DCG, and AK = CG. then erect KL perpendicular to the plane HAI, and let KL be = GF. & draw AL: then AHIL ſhall be a ſolid angle equall to that given CDEF. For the conſtruction of this do's wholly reſemble the framing of that, as may eaſily appear to any that examine it.

Prop.

PROP. XXVII.

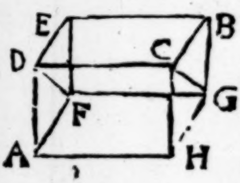


Upon a right line given AB *to describe a parallelepipedon* AK, *like, and in like manner situate, with a solid parallelepipedon given* CD.

Of the plane angles, BAH, HAI, BAI, which are equall to FCE, ECG, FCG, *a* make the solid angle A equall to the solid angle C. also *b* make FC. CE :: BA. AH. *b* and CE. CG :: AH. AI (*c* whence of equality FC. CG :: BA. AI) and finish the parallelepipedon AK, which shall be like to that which is given. a 26. 11. b 12. 6. c 22. 5.

For by the construction, the Pgr. *d* BH is like to FE, and *d* HI to EG, and *d* BI to FG, & *e* so the opposites of these to the opposites of them: therefore the six planes of the solid AK are like to the six planes of the solid CD, *f* and consequently AK, CD, are like solids. *W.W. to be Done.* d 1. def. 6. e 24. 11. f 9. def. 11.

PROP. XXVIII.

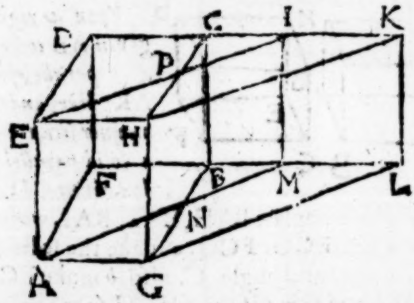


If a solid parallelepipedon AB *be cut by a plane* FGCD *drawn by the diagonall lines* DF, CG, *of the opposite planes* AE, HB, *that solid* AB *shall be equally bisected by the plane* FGCD.

For because DC, FG, are *a* equall and parallels, *b* the plane FGCD is a Pgr. and being *a* the Pgrs. AE, HB, are equall and like, *b* also the triangles AFD, HGC, CGB, DFE are equall and like. But the Pgrs. AC, AG, are equall and like to FB and FD. therefore all the planes of the prisme FGCDAH are equall and like to all the planes of the prisme FGCDEB, and *c* consequently this prisme is equall to that. *W.W. to be Dem.* a 24. 11. b 34. 1. c 9. def. 11.

PROP.

PROP. XXIX.

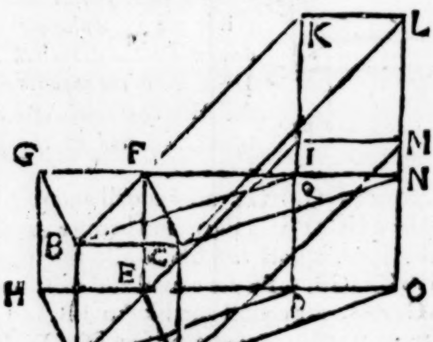


Solid parallelepipedons AGHEFBCD, AGHEMLKI, *being constituted upon the same base* AGHE, *and* * *in the same height, whose insisting lines* AF, AM, *are placed in the same right lines* AG, FL, *are equall one to the other.*

* *i. e. between the parallel planes* AGHE, FLKD. *and so understand it in the foll.*
a 10. *def.* 11. *and* 35. 1.
b 3. *and* 2. *ax.* 1.

For a if from the equall prismes AFMEDI, GBLHCK, the common prisme NBMPCI be taken away, and the solid AGNEHR be added, the Parallelepiped. AGHEFBCD shall be = AGHEMLKI. *W.W. to be Dem.*

PROP. XXX.

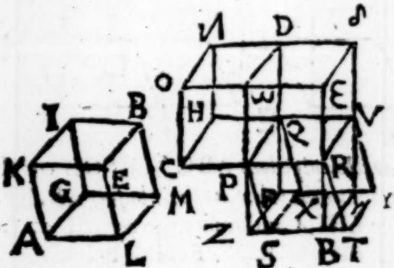


Solid parallelepipedons ADBCHEFG, ADGBIMLK *being*

being constituted upon the same base ADEC, *and in the same height, whose insisting lines* AH, AI *are not placed in the same right lines, are equall one to the other.*

For produce the right lines HEO, GFN, and LMO, KIP; and draw AP, DO, BQ, CN. a then shall DC, AB, HG, EF, PQ, ON be as well equall and parallel one to the other as AD, HE, GF, BC, KL, IM, QN, PO. b wherefore the parallelepipedon ADCBPONQ shall be equall to either parallelepipedon ADCBHEFG, ADCBIMLK; and c consequently these two are equall one to the other. *W. W. to be Dem.* a 34. 1. b 29. 11. c 1. ax. 1.

P r o p. XXXI.



Solid parallelepipedons, ALEKGMBI, CPαOHQDN, *being constituted upon equall bases* ALEK, CPαO, *and * in the same height are equall, one to the other.*

* *by height understand the perpendicular drawn from the plane of the base to the opposite plane.*

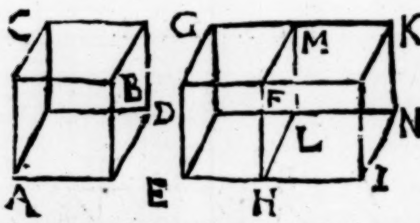
First, let the parallelepipedons AB, CD, have the sides perpendicular to the bases. and at the side CP being produced, a make the Pgr. PRTS equall and like to the pgr. KELA. b and so the parallelepipedon PRTSQVYX equall and like to the parallelepipedon AB. Produce OαE, NDδ, αPZ, DQF, ERB, δVγ, TSZ, YXF; and draw Eδ, Bγ, ZF. a 18. 6. b 27. 11. and 10 def. 11.

The planes OδN, CRVH, ZTYF. c are parallels one to the other; d and the Pgrs. ALEK, CDαO, PRTS, PRBZ are equall. Therefore since the parallelepipedon CD. PVδα :: Pgr. Cα (PRBZ) Pr e :: pa- c 30. def. 11. d hyp. and 35. 1. e 25. 11.

parallelepipedon PRBZQVγF. PV$\delta\omega$; the parallelepipedon CDf shall be = PRBZQVγF g = PR.
f 9. 5. g 29. 11. h constr. VQSTYX h = AB. *W.W. to be Dem.*

But if the parallelepipedons AB, CD, have sides oblique to the base, then on the same bases and in the same height place parallelepipedons whose sides
k 29. 11. m 1. ax. 1. are perpendicular to the base. k They shall be equall to one another, and those that are oblique m whence also the oblique parallelepipedons AB, CD are equall. *W.W. to be Dem.*

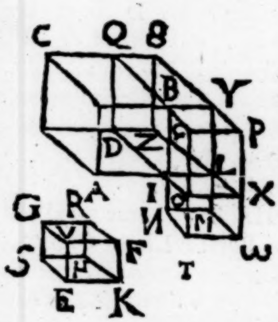
Prop. XXXII.



Solid parallelepipedons ABCD, EFGL, *of the same height, are one to the other, as their bases,* AB, EF.

Produce EHI, a and make the pgr. FI = AB, and
b 31. 11. b compleat the parallepp. FINM. It is clear that the
c 31. 11. parallepp. FINM. (c ABCD.) EFGL d :: FI (AB.)
d 25. 11. EF. *W.W. to be Dem.*

Prop. XXXIII.



Like solid parallelepipedons, ABCD, EFGH, *are in tripled proportion one to the other, of that in which their homologous sides or of like proportion* AI, EK, *are.*

Produce the right lines AIL, DIO, BIN, and
a 3. 1. a make IL, IO, IN, equall

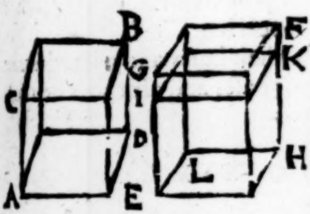
to

to EK, KH, KF, *b* and so the parallepp. IXMT equall b 27. 11.
and like to the parallepp. EFGH. *c* Let the pa- c 31. 1.
rallepps. IXPB, DLYQ be finished. *d* Then shall be d *hyp.*
AI.IL (EK) :: DI.IO (HK) :: BI.IN.KF. *e* that is e 1. 6.
the Pgr. AD.DL :: DL.IX :: BO.IT. *f i.e.* the paral- f 32. 11.
lepp. ABCD. DLQY :: DLQY. IXBP :: IXBP.IX-
MT (*g* EFGH.) *h* therefore the proportion of ABCD g *constr.* h 10. *def.* 5.
to EFGH is triple of the proportion of ABCD to k 1. 6.
DLQY, *k* or of AI to EK. *W.W. to be Dem.*

Coroll.

Hence it appears that if four right lines be continually proportionall, as the first is to the fourth, so is a parallelepipedon described on the first to a parallelepipedon described on the second, being like and in like manner described.

PROP. XXXIV.



In equall solid parallelepipedons AD-CB, EHGF, *the bases and altitudes are reciprocall* (AD. EH :: EG. AC.) *And solid parallelepipedons,* AD-CB, EHGF, *whose bases and altitudes are reciprocall, are equall.*

First, let the sides CB, GE be perpendicular to the bases; then if the altitudes of the solids are equall, the bases also shall be equall. and the thing is clear. But if the altitudes are unequall, from the greater EG *a* take EI = AC, and at I *b* draw the plane IK parallel to the base EH. then

a 3. 11. b 31. 1. c 32. 11.

1. *Hyp.* AD.EH *c* :: parallepp. ADCB.EHIK *d* :: parallepp. EHGF. EHIK *e* :: GL. IL *e* :: GE. IE. (*f* AC.) *g* it is plain therefore that AD.EH :: GE.AC. *W.W. to be Dem.*

d 7. 5. e 1. 6. f *constr.* g 11. 5. h 32. 11

2. *Hyp.* ADCB.EHIK *h* :: AD.EH *k* :: EG.EI *l* :: GL. IL *m* :: parallepp. EHGF. EHIK. *n* where-fore

k *hyp.* l 1. 6. m 32. 11. n 9. 5.

fore the parallelepipedon ADCB = EHGF. *W.W. to be Dem.*

Moreover, let the sides be oblique to the bases, and erect right parallelepipedons upon the same bases in the same altitude; the oblique parallepps shall be equall to them. Wherefore since by the first part, the bases and altitudes of those be reciprocall, the bases and altitudes of these also shall be reciprocall. *W.W. to be Dem.*

Coroll.

All that hath been demonstrated of parallelepipedons in the 29, 30, 31, 32, 33, 34. Prop. *does also agree to triangular prismes, which are half parallelepipedons, as appears by Prop.* 28. Therefore,

1. Triangular prismes are of equal height with their bases.

2. If they have the same or equall bases and the same altitude, they are equall.

3. If they be like, their proportion is treble to that of their sides of like proportion.

4. If they be equall, their bases and altitudes are reciprocall; and if their bases and altitudes be reciprocall, they are also equall.

PROP. XXXV.

If there be two plane angles BAC, EDF, *equall, and from the points of those angles two right lines* AG, DH, *be elevated on high, conteining equall angles with the lines first given, each to his correspondent angle (the angle* GAB = HDE, *and* GAC = HDF.) *and if in those elevated lines* AG, DH, *some points be taken,* G, H; *and from these points perpendicular lines* GI, HK, *drawn to the planes* BAC, EDF, *in which the angles first given are, and right lines*

lines AI, DK, *be drawn to the angles first given from the points* I, K, *which are made by the perpendiculars in the planes; those right lines with the elevated lines* AG, DH *shall contain equall angles* GAM, HDK.

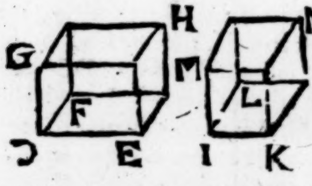
Make DH, AL, equall; and GI, LM parallels. and MC to AC, MB to AB, KF to DF, KE to DE perpendicular; and draw the right lines BC, LB, LC, and EF, HF, HE; [a] and LM is perpendicular to the plane BAC; [b] wherefore the angles LMC, LMA, LMB; and by the same reason the angles HKF, HKD, HKE are right angles. Therefore ALq [c] = LMq + AMq [c] = LMq + CMq + ACq [c] = LCq + ACq. [d] therefore the angle ACL is a right angle. Again ALq [e] = LMq + MAq [e] = LMq + BMq + BAq [e] = BLq + BAq. [d] therefore the angle ABL is also a right angle. By the like inference the angles DFH, DEH are right angles; [f] therefore AB = DE, [f] and BL = EH, [f] and AC = DF, and CL = FH. [g] wherefore also BC = EF; [g] and the angle ABC = DEF, [g] and the angle ACB = DFE. [h] whence the other right angles CBM, BCM, are equall to the other FEK, EFK. [k] therefore CM = FK, [l] and so also AM = DK. therefore if from LAq [m] = HDq be taken away AMq = DKq, [n] there remains LMq = HKq. wherefore the triangles LAM, HDK are equilaterall one to the other; [o] therefore the angle LAM = HDK. *W.W. to be Dem.*

a 8. 11.
b 3. def. 11.
c 47. 1.
d 48. 1.
e 47. 1.
f 26. 1.
g 4. 1.
h 1. ax. 1.
k 16. 1.
l 47. 1.
m *constr.*
n 47. 1 *and* 3. *ax.*
o 8. 1.

Coroll.

Therefore, if there be two plane angles equall, from whose points equall right lines be elevated on high containing equall angles with the lines first given, each to each; perpendiculars drawn from the extreme points of those elevated lines to the planes of the angles first given, are equall one to the other; viz. LM = HK.

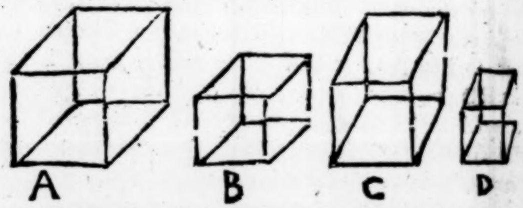
Prop. XXXVI.



If there be three right lines DE, DG, DF *proportionall, the solid parallelepipedon* DH *made of them, is equall to the solid parallelepipedon* IN *made of the middle line* DG (*IL*) *which is also equilaterall, and equiangular to the said parallelepp.* DH.

a hyp. b 14. 6. Because DE.IK a :: IL.DF. b the pgr. LK shall be = FE. and by reason of the equality of the plane angles at E and I, and of the lines GD, IM, also the altitudes of the parallelepipedons are equall by the prec. Coroll. c therefore the parallepps are equall one to the other. *W.W. to be Dem.* c 31, 11.

Prop. XXXVII.



If there be four right lines A, B. C, D, *proportionall, the solid parallelepipedons* A, B, C, D *being like, and in like sort described from them, shall be proportionall. And if the solid parallelepipedons, being like and in like sort described be proportionall* (A. B :: C. D.) *then those right lines* A, B, C, D, *shall be proportionall.*

a 33. 11. b sch. 23. 5. For the proportions of the parallelepps. a are triple of those of the lines; therefore if A. B :: C. D. b then shall the parallepp. A. parallepp. B :: parallepp. C. parallepp. D. and so also contrarily.

PROP.

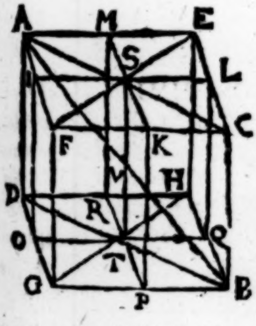
PROP. XXXVIII.

If a plane AB *be perpendicular to a plane* AC, *and a perpendicular line* EF *drawn from a point* E *in one of the planes* (AB) *to the other plane* AC, *that perpendicular* EF *shall fall upon the common section of the planes* AD.

If it be possible, let F fall without the intersection AD. and in the plane AC *a* draw FG perpendicular to AD, and join EG. The angle FGE *b* is a right angle, and EFG is supposed to be such also; therefore two right angles are in the triangle EFG. *c Which is Absurd.*

a 12. 1.
b 4. and 3. def. 11.
c 17. 1.

PROP. XXXIX.



If the sides (AE, FC, AF, EC, *and* DH, GB, DG, HB) *of the opposite planes* AC, DB, *of a solid parallelepipedon* AB, *be divided into two equall parts, & planes* ILQO, PKMR, *be drawn through their sections, the common section of the planes* ST, *and the diameter of the solid parallelepipedon* AB *shall divide one the other into two equall parts.*

Draw the right lines SA, SC, TD, TB. Because *a* the sides DO, OT are equall to the sides BQ, QT, *b* and the alternate angles TOD, TQB equall also, *c* the bases DT, TB, & the angles DTO, BTQ are equall. *d* therefore DTB is a right line. & so in like manner is ASC. Moreover *e* as well AD is parallel & equall to FG *e* as FG to CB, & *f* thence AD is parallel & equal to CB; *g* & consequently AC to DB. *h* wherefore AB

a 34. 1.
b 29. 1.
c 4. 1.
d sch. 15. 1.
e 34. 1.
f 9. 11. and 1. ax.
g 33. 1.
h 7. 11.

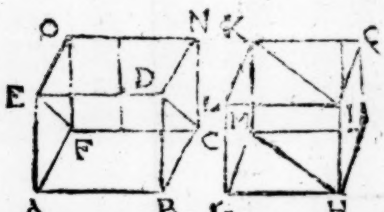
and ST are in the same plane ABCD. Therefore since the verticall angles AVS, BVT, & the alternate angles ASV, BTV are equall; k and AS = BT; therefore shall AV be = BV, l and SV = VT. *W.W. to be Dem.*

k 7 ax. 1.
l 26. 1.

Coroll.

Hence in every parallelepipedon all the diameters bisect one another in one point, V.

Prop. XL.



If two prismes ABCFED, GHMLIK, *be of equall altitude, whereof one hath its base* ABCF *a parallelogram, and the other* GHM *a triangle; and if the parallelogram* ABCF *be double to the triangle* GHM; *those prismes* ABCFED, GHMLIK *are equall.*

For if the parallelepps. AN, GQ, be completed, a they shall be equall because of the equality b of the bases AC, GP, & c of the altitudes. d therefore also the prismes, e the halfes thereof shall be equall. *W.W. to be Dem.*

a 31. 11.
b 34. 1. *and* 7. *ax.*
c *hyp.*
d 28. 11.
e 7 ax. 1.

Schol.

From the preceding demonstrations, the dimension of triangular prismes, and quadrangular, or parallelepipedons, is learnt; viz. *by multiplying the altitude into the base.*

Andr. Tacq.

As if the altitude be 10 foot, and the base 100 square foot (the base may be measured by *sch.* 35. 1. or by 41. 1.) then multiply 100 by 10. and 1000 cubic

cubic foot ſhall be produced for the ſolidity of the priſme given.

For as a rectangle, ſo alſo is a right parallelepipedon produced of the altitude multiplyed into the baſe. Therefore every parallelepipedon is produced of the altitude multiplyed into the baſe, as appears by 31. of this Book.

Moreover, ſince the whole parallelepipedon is produced of the altitude drawn into the baſe, the half thereof (that is, a triangular priſme) ſhall be produced of the altitude drawn into half the baſe, namely the triangle.

An Advertiſement.

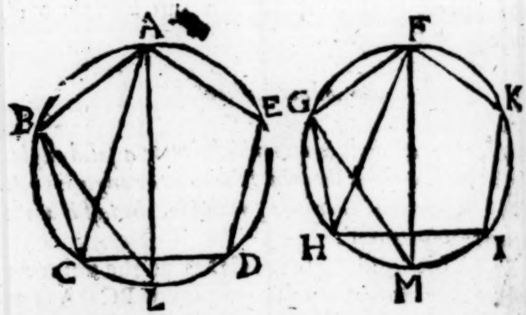
Obſ. That of thoſe letters which denote a ſolid angle, the firſt is alwayes at the point in which the angle is; but of thoſe letters which denote a pyramide, the laſt is at the ſupreme point thereof.

Ex. gr. the ſolid angle ABCD is at the point A; and the ſupreme point of the pyramide BCDA is at the point A. and the baſe is the triangle BCD.

The End of the eleventh Book.

THE TWELFTH BOOK OF EUCLIDE'S ELEMENTS.

PROPOSITION I.



Like polygonous figures ABCDE, FGHIK, *described in circles* ABD, FGI, *are one to another, as the squares described of the diameters of the circles* AL, FM.

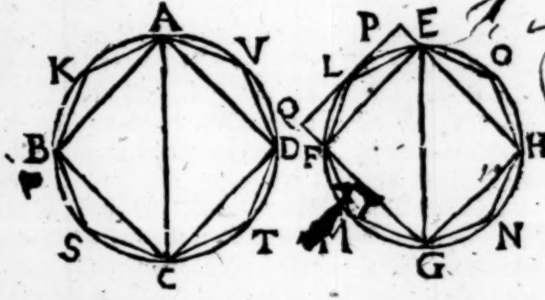
Draw AC, BL, FH, GM. Because
a 1 def. 6. *a* the angle ABC = FGH, *a* and AB.BC :: FG.GH.
b 6. 6. *b* therefore shall the angle ACB (*c* ALB) be = FHG
c 21. 3. (*c* FMG.) but the angles ABL, FGM *d* are right
d 31. 3. and so equall; *e* therefore the triangles ABL, FGM
e 32. 3. are equiangular. *f* wherefore AB. FG :: AL. FM.
f cor. 4. 6. *g* therefore ABCDE. FGHIK :: ALq.FMq.
g 22. 6.

Coroll.

Hence (because AB. FG :: AL. FM :: BC. GH, &c.) the contents of like polygonous figures described in a circle are in *h* proportion as the diameters.
h 1. 12. and 12. 5.

PROP.

PROP. II.



Circles ABT, EFN, *are in proportion one to another, as the squares of their diameters* AC, EG *are.*

Suppose ACq. EGq :: the circle ABT. I. I say then I is equall to the circle EFN.

For first, if it be possible, let I be lesse then the circle EFN, and let K be the excesse or difference. Inscribe the square EFGH in the circle EFN, [a] it being the half of a circumscribed square, and so greater then the semicircle. [b] Divide equally in two the arches EF, FG, GH, HE, and at the points of the divisions join the right lines EL, LF, &c. at L draw the tangent PQ ([c] which is parallel to EF) and produce HEP, GFQ. then is the triangle ELF [d] the half of the pgr. EPQF, and so greater then the half of the segment ELF; and in like sort the rest of those triangles exceed the halfs of the rest of the segments. And if the arches EL, LF, FM, &c. be again bisected, and the right lines joined, the triangles will likewise exceed the half of the segments. Wherefore if the square EFGH be taken from the circle EFN, and the triangles from the other segments, and this be done continually, at length [e] there will remain some magnitude lesse then K. Let us have gone so farre, namely to the segments EL, LF, FM, &c. taken together

a *sch.* 7. 4.
b 30. 3.
c *sch.* 27. 3.
d 41. 1.
e 1. 10.

f hyp. and 3. ax. gether lesse then K. Therefore I (*f* the circle EFN − K) $\sqsupset$ the polyg. ELFMNHO (the circle EFN −
g 30. 3. and 1. post. 1. the segm. EL + LF, &c.) In the circle ABT *g* conceive a like polygonon AKBSCTDV inscribed.
h 1. 12. therefore since AKBSCTDV. ELFMGNHO *h* ::
k hyp. ACq. EGq *k* :: the circle ABT. I. and the polyg.
l 9. ax. 1. AKBSCTDV *l* $\sqsupset$ the circle ABT. the polyg. ELFM-
m 14. 5. GNHO *m* shall be $\sqsupset$ I. but before, I was $\sqsupset$ ELFMGNHO. which is repugnant.

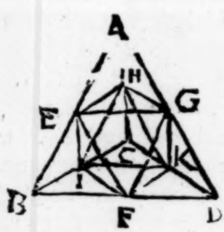
Again, if it be possible, let I be $\sqsubset$ the circle EFN.
n hyp. Therefore because ACq. EGq *n* :: the circle ABT. I; and inversely I. the circle ABT :: EGq. ACq. sup-
o 14. 5. pose I. the circle ABT :: the circle EFN. K. *o* there-
p 11. 5. fore the circle ABT $\sqsubset$ K. *p* and EGq. ACq :: the circle EFN. K. which is shewn to be repugnant.

Therefore it must be concluded, that I is = to the circle EFN. *W.W. to be Dem.*

Coroll.

Hence it follows, that as a circle is to a circle, so is a polygonon described in one to a like polygonon described in the other.

Prop. III.



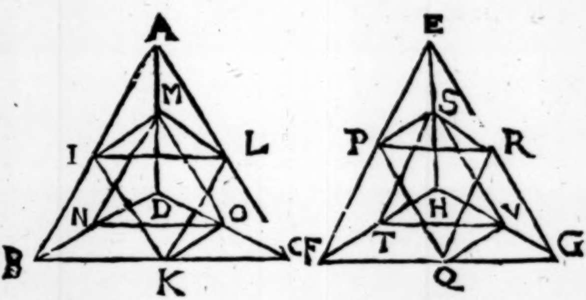
Every Pyramide ABDC *having a triangular base, may be divided into two pyramides* AEGH, HIKC, *equall, and like one to the other, having bases triangular, and like to the whole* ABDC; *and into two equall prismes*, BFGEIH, EGDIHK; *which two prismes are greater then the half of the whole pyramide* ABDC.

Divide the sides of the pyramide into two parts at the points E, F, G, H, I, K. and join the right lines EF, FG, GE, EI, IF, FK, KG, GH, HE. Because the sides of the pyramide are proportionally cut,

cut, *a* thence HI, AB; and GF, AB; and IF, DC; and HG, DC, &c. are parallels. and consequently HI, FG; and GH, FI are also parallels. therefore it is apparent that the triangles ABD, AEG, EBF, FDG, HIK, *b* are equiangular. and that the four last are *c* equall: in like manner the triangles ACB, AHE, EIB, HIC, FGK are equiangular; and the four last are equall one to the other. Also the triangles BFI, FDK, IKC, EGH; & lastly the triangles AHG, GDK, HKC, EFI are like and equall. Moreover the triangles, HIK to ADB, and EGH to BDC, and EFI to ADC, and FGK to ABC, *d* are parallel. From whence it evidently follows, first that the pyramids AEGH, HIKC are equall, and *e* like to the whole ABDC, and to one another. Next, that the solids BFGEIH, FGDIHK are prismes, and that of equall height, as being placed between the parallel planes ABD, HIK. but the base BFGE is *f* double of the base FDG. wherefore the said prismes are *g* equall; whereof the one BFGEIH is greater then the pyramide BEFI, that is, then AEGH, the whole then its part; & consequently the two prismes are greater then the two pyramids and so exceed the half of the whole pyramide ABDC. *W.W. to be Dem.*

a 2.6.
b 29.1.
c 26.1.
d 15.11.
e 10.def.11.
f 2.ax.1.
g 40.11.

PROP. IV.



If there be two pyramids ABCD, EFGH, *of the same altitude, having triangular bases* ABC, EFG; *and*

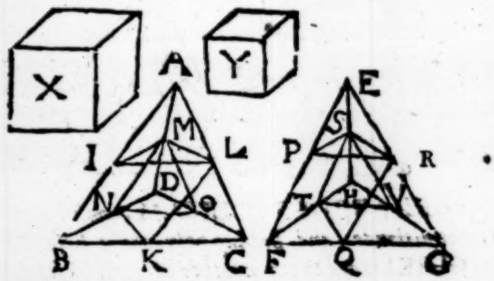
*and either of them be divided into two pyramids (*AILM, MNOD *; and* EPRS, STVH) *equall one to the other and like to the whole , and into two equall prismes* (IBKLMN, KLCNMO; *and* PFQRST , QRGTSV;) *and if in like manner either of those pyramids made by the former division be divided , and this be done continually; then as the base of one pyramide is to the base of the other pyramide, so are all the prismes which are in one pyramide , to all the prismes which are in the other pyramide, being equall in multitude.*

For (applying the construction of the precedent
a 15. 5. *prop.*) BC. KC *a* :: FG. QG. *b* therefore the triangle
b 22. 6. ABC is to the like triangle LKC as EFG is to *c* the
c 2. 6. *&c.* like RQG. therefore by permutation ABC. EFG *d* ::
d 16. 5. LKC. RQG *e* :: the prisme KLCNMO. QRGTSV
e *sch.* 34. 11. (for these are of equall altitude) *f* :: IBKLMN.
f 7. 5. PFQRST. *g* wherefore the triang. ABC. EFG :: the
g 12. 5. prism. KLCMNO + IBKLMN. the prisme QRGTSV + PFQRST. *W.W. to be Dem.*

But if the pyramids MNOD, AILM ; and EPRS, STVH ; be further divided in like manner the four new prismes made hereby shall be to the four produced before as the bases MNO and AIL are to the bases STV, and EPR; that is, as LKC to RQG, or as ABC to EFG. *b* wherefore all the prismes of the pyramide ABCD are to all the prismes of the pyramide EFGH as the base ABC is to the base EFG. *W.W. to be Dem.*

PROP.

PROP. V.



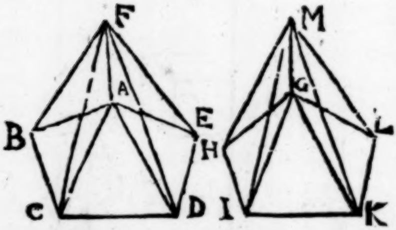
Pyramids ABCD, EFGH, *being under the same altitude, having triangular bases* ABC, EFG, *are one to another as their bases* ABC, EFG, *are.*

Let the triangle ABC. EFG :: ABCD. X. I say X is equall to the pyramide EFGH. For if it be possible, let X be $\sqsupset$ EFGH. and let the excesse be Y. Divide the pyramide EFGH into prismes and pyramids, and the other pyramids in like manner, [a] till the pyramids left EPRS, STVH, be lesse then the solid Y. Therefore since the pyramide EFGH = X + Y, it is manifest that the remaining prismes PFQRST, QRGTSV are greater then the solid X. Conceive the pyramide ABCD divided after the same manner; [b] then will be the prisme IBKLMN + KLCNMO. PFQRST + QRGTSV :: ABC. EFG [c] :: the pyr. ABCD. X. [d] therefore X $\sqsubset$ the prisme PFQRST + QRGTSV; which is contrary to that which was affirmed before. a 11 10. b 4 12. c hyp. d 14 5.

Again, conceive X $\sqsubset$ the pyr. EFGH. and make the pyr. EFGH. Y :: X. the pyr. ABCD [e] :: EFG. ABC. Because EFGH [f] $\sqsupset$ X, [g] thence Y $\sqsupset$ the pyr. ABCD. which is shewn before to be impossible. Therefore I conclude, that X is equall to the pyr. EFGH. *W. W. to be Dem.* e hyp. and cor. 4 5. f suppos. g 14 5.

PROP.

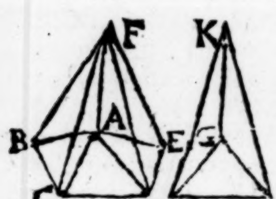
PROP. VI.



Pyramids ABCDEF , GHIKLM , *consisting under the same altitude , and having polygonous bases* ABCDE, GHIKL , *are to one another as their bases* ABCDE,FGHIKL *are.*

Draw the right lines AC,AD,GI,GK. then is the base ABC.ACD *a* :: the pyr. ABCF.ACDF. *b* therefore by composition, ABCD.ACD :: the pyr. ABCDF.ACDF. *a* but also ACD.ADE :: the pyr. ACDF. ADEF. *c* therefore of equality ABCD. ADE :: ABCDF. ADEF. and *b* thence by composition ABCDE.ADE :: the pyr. ABCDEF. ADEF. moreover ADE. GKL *d* :: the pyr. ADEF. GKLM; and, as before , and inversely GKL. GHIKL :: the pyr. GKLM. GHIKLM. *c* therefore again of equality ABCDE. GHIKL :: the pyr. ABCDEF. GHIKLM. *W. W. to be Dem.*

a 5. 12.
b 18. 5.
c 22. 5.
d 5. 12.

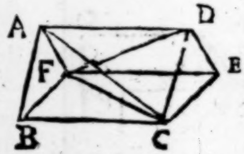


If the bases have not sides of equall multitude, the demonstration will proceed thus. The base ABC.GHI *e* :: the pyr. ABCF. GHIK. *e* & ACD. GHI :: the pyr. ACDF. GHIK. *f* therefore the base ABCD. GHI :: the pyr. ABCDF. GHIK. *e* Moreover the base ADE.GHI :: the pyr. ADEF.GHIK. *f* therefore the base ABCDE. GHI :: the pyr. ABCDEF. GHIK.

e 5. 12.
f 24. 5.

PROP.

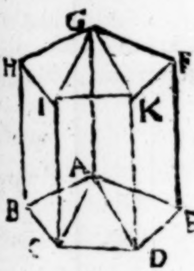
PROP. VII.



Every prisme , ABCDEF. *having a triangular base, may be divided into three pyramids* ACBF , ACDF , CDFE, *equall one to the other, & having triangular bases.*

Draw the diameters of the parallelograms , AC, CF, FD. Then the triangle ACB is *a* = ACD. *b* therefore the pyramids of equall height ACBF, ACDF. are equall. In like manner the pyr. DFAC = the pyr. DFEC. but ACDF and DFAC are one & the same pyramide. *c* therefore the three pyramids ACBF, ACDF, DFEC , into which the prisme is divided, are equall one to the other. *W.W. to be Dem.*

a 34.1. b 5.11. c 1.ax.1.



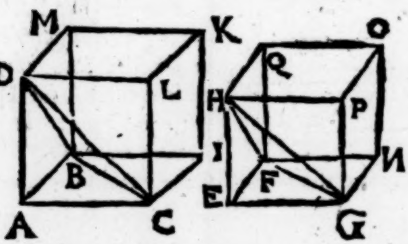
Hence , every pyramide is the third part of the prisme that ha's the same base and height with it; or every prisme is treble of the pyramide that ha's the same base and height with it.

For resolve the polygonous prisme ABCDEGHIKF into triangular prismes ; and the pyramide ABCDEH into triangular pyramides ; *a* then all the parts of the prisme shall be treble to all the parts of the pyramide, *b* consequently the whole prisme ABCDEGHIKF is treble to the whole pyramide ABCDEH. *W. W. to be Dem.*

a 7.11. b 1.5.

PROP.

PROP. VIII.



Like pyramids ABCD, EFGH, *which have triangular bases* ABC, EFG *are in triple proportion of that in which their sides of like proportion* AC, EG, *are*.

a 27. 11. b 9 *def.* 11. c 28. 11. *and* 7. 12. d 15. 5. e 33. 11.

a Complete the parallelepipedons ABICDMKL, EFNGHQOP, which b are like, and c sextuple of the pyramids ABCD, EFGH. d and therefore in the same proportion with them one to another, e that is, triple of that of the sides of like proportion, &c.

Coroll.

Hence, also like polygonous pyramids have proportion tripled to that of the sides of like proportion; as may easily be proved by resolving the same into triangular pyramids.

PROP. IX.

See the prec. Scheme.

In equall pyramids ABCD, EFGH, *having triangular bases* ABC, EFG, *the bases and altitudes are reciprocall; And pyramids having triangular bases, whose altitudes and bases are reciprocall, are equall.*

1. *Hyp.* The completed parallelepipedons ABICDMKL, EFNGHQOP are a sextuple of the equall pyramids ABCD, EFGH (either of either) and so equall

a 28. 11. *and* 7. 12.

equall one to the other. therefore the altitude (H.) the alt. (D) b :: ABIC. EFNG c :: ABC. EFG. *W.W. to be Dem.* b 34. 11. c 15. 5.

2. *Hyp.* The altitude (H.) the alt. (D) d :: ABC. EFG e :: ABIC. EFNG. f therefore the parallelepipedons ABICDMKL, EFNGHQOP are equall. g consequently also the pyramids ABCD, EFGH being subsextuple of the same, are equall. *W.W. to be Dem.* d hyp. e 15. 5. f 34. 11. g 6. ax. 1.

The same is applicable to polygonous pyramids; for they may also in like manner be reduced to triangulars.

Coroll.

Whatsoever is demonstrated of pyramids in prop. 6, 8, 9. *do's likewise agree to any sort of prismes; seeing they are triple of the pyramids that have the same base and altitude with them.* Therefore.

1. The proportion of prismes of equall altitude is the same with that of their bases.

2. The proportion of like prismes is triple of that of the sides of like proportion.

3. Equall prismes have their bases and altitudes reciprocall; and prismes which are so reciprocall, are equall.

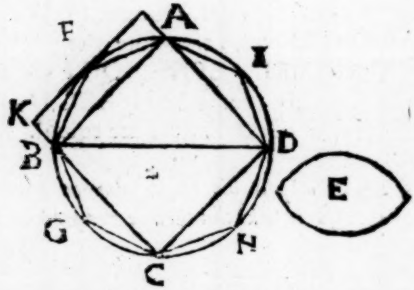
Schol.

From what is hitherto demonstrated the dimension of any prismes and pyramids may be collected.

a The soliditie of a prisme is produced of the altitude multiplyed into the base; b and therefore likewise that of a pyramide, of the third part of the altitude multiplyed into the base. a 1. cor 12. and sch. 40. 11. b 7. 12.

V PROP.

PROP. X.



Every Cone is the the third part of a cylinder having the same base with it ABCD, *and the altitude equall.*

See the second fig. of this Book.

If you deny it, then first let such cylinder be more then triple to the cone, and let the excesse be E. A prisme described on a square in the circle ABCD *a* is subduple of a prisme described upon a square about the circle, being equall to it and the cylinder in height. Therefore a prisme upon the square ABCD exceeds the half of the cylinder. and likewise a prisme upon the base AFB, of equall height to the cylinder, *b* is greater then the half of the segment of the cylinder AFB. continue an equall bisection of the arches, & substract the prismes till the remaining segments of the cylinder, namely at AF, FB, &c. become less then the solid E. Therefore the cyl.— segm. AF, FB, &c. (the prisme on the base AFBGCHDI) *c* is greater then the cylinder — E (*d* the triple of the cone.) therefore the pyramide, *e* a third part of the said prisme (being placed on the same base, and of the same height) is greater then the cone of equall height on the base ABCD a circle, *i. e.* the part greater then the whole *Which is Absurd.*

a sch 7.4. and cor 9.12.

b sch. 17.3. and cor. 9. 12.

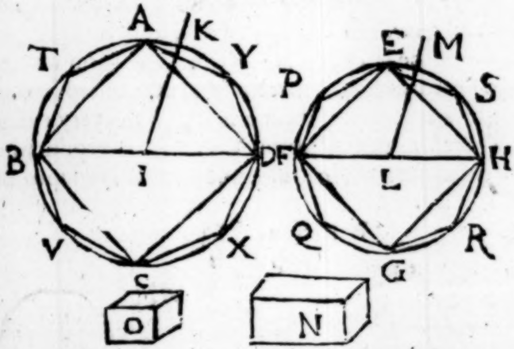
c 5. ax. 1.
d hyp.
e cor. 7. 12.

But if the cone be affirmed to be greater then the third part of the cylinder, then let the excesse be E. Detract the pyramids from the cone, as you did in the first part the prismes from the cylinder, till some

segments

ſegments of the cone remain, conceive at AF, FB, BG,&c. leſſe then the ſolid E. therefore the cone — E (*f* $\frac{1}{3}$ of the cylin.) ⊐ the pyr. AFBGCHDI (the cone — ſegm. AF, FB, &c.) therefore the priſme triple to the pyramide (*viz.* of equall height, and on the ſame baſe) is greater then the cylinder on the baſe ABCD, the part then the whole. *Which is Abſ.* Wherefore it muſt be granted, that the cylinder is equall to triple of the cone. *W.W. to be Dem.* *f hyp.*

PROP. XI.



Cylinders and Cones ABCDK, EFGHM, *being under the ſame altitude, are to one another as their baſes* ABCD, EFGH *are.*

Let the circle ABCD. the cir. EFGH :: the cone ABCDK. N. I ſay N is equall to the cone EFGHM.

For if it be poſſible, let N be ⊐ the cone EFGHM, and let the exceſſe be O. The preparation and argumentation of the prec. prop. being ſuppoſed; then ſhal O be greater then the ſegments of the cone EP, PF, FQ, &c. and ſo the ſolid N ⊐ the pyr. EPFQGRHSM. In the circle ABCD *a* make a like polygonous fig. ATBVCXDY. Becauſe the pyr. ABVYK. the pyr. EFQSM *b* :: the polyg. ATBVY. the polyg. EPFQS *c* :: the cir. ABCD. the cir. EFGH *d* :: the cone ABCDK. N. *e* thence the pyr. EPFQGRHSM ſhall be ⊐ N. contrary to what was affirmed before. Again conceive N ⊏ the cone EFGHM. and

a 303 *and* 1. *poſt.*
b 6. 12.
c cor. 2 12.
d hyp.
e 14 5.

& make the cone EFGHM. O :: N. the cone ABCDK *f* :: the circ. EFGH. ABCD. *g* therefore O ⊐ the cone ABCDK; *which is absurd*, as appears by by what is shewn in the first part.

f hyp. and by inversion.
g 14. 5.

Therefore rather admit ABCD. EFGH :: the cone ABCDK. EFGHM. *W. W. to be Dem.*

The same may be demonstrated of cylinders, if cylinders and prismes be conceived in the place of cones and pyramids. therefore, &c.

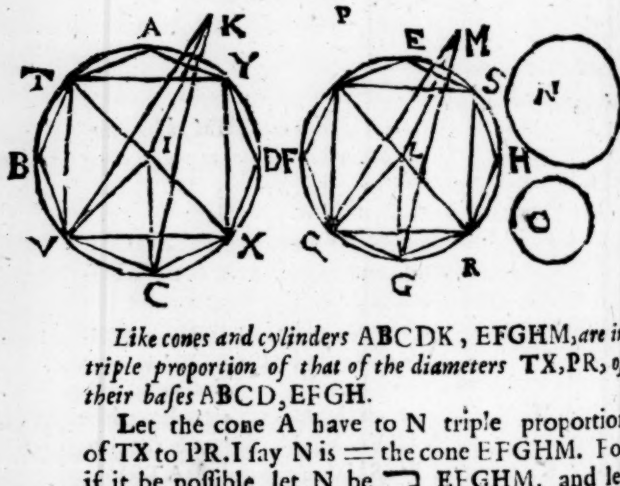
Schol.

Hence, is gathered the dimension of all sorts of cylinders and cones. The soliditie of a right cylinder is produced of the circular base (*a* the dimension whereof is to be learnt out of *Archimedes*) multiplied into the height; *b* whence in like manner that of every cylind.

a 1. Prop. de dimens. circ
b 11. 12.

Therefore the soliditie of a cone is produced of the third part of the altitude multiplyed into the base.

PROP. XII.



Like cones and cylinders ABCDK, EFGHM, *are in triple proportion of that of the diameters* TX, PR, *of their bases* ABCD, EFGH.

Let the cone A have to N triple proportion of TX to PR. I say N is = the cone EFGHM. For if it be possible let N be ⊐ EFGHM. and let the excesse be O. therefore, N ⊐ the pyr. EPFQGRHSM. Let the axes of the cones be IK, LM, and join the right lines VK, CK, VI, CI, and QM,

QM, GM, QL, GL. Because the cones are like, [a] thence VI. IK :: QL. LM. but the angles VIK, QLM [b] are right angles. [c] therefore the triangles VIK, QLM are equiangul. [d] whence VC. VI :: QG. QL. also VI. VK :: QL. QM. therefore of equality VC. VK :: QG. QM. [e] moreover VK. CK :: QM. MG. therefore again of equality VC. CK :: QG. GM. [f] therefore the triangles VKC, QMG are like: and by the same reason the other triangles of this pyramide are like to the other of that. [g] wherefore the pyramids themselves are like. [h] But they are in triple proportion of that of VC to QG, [k] that is, of VI to RL, [l] or TX to PR. [m] therefore the pyr. AIBVCXDYK. the pyr. EPFQGRHSM :: the cone ABCDK. N. [n] whence the pyr. EPFQGRHSM $\sqsupset$ N. which is repugnant to what was affirmed before.

a 14 *def.* 11.
b 18. *def.* 11.
c 6. 6.
d 4. 6.
e 7. 5.
f 5. 6.
g 9. *def.* 11.
h *cor.* 8. 12.
k 4. 6.
l 15. 5.
m *hyp.* and 11. 5.
n 14. 5.

Again, take N $\sqsubset$ the cone EFGHM. make the cone EFGHM. O :: N. the cone ABCDK. [o] :: the pyr. EPRM. ATCK [p] :: GQ. VC thrice :: [q] PR. TX thrice. but O [r] is $\sqsupset$ ABCDK. which was before shewn to be repugnant. Wherefore N = the cone EFGHM. *W. W. to be Dem.*

o *before.* & *inversely.*
p *cor.* 8. 12.
q 4. 6.
r 14. 5.

But forasmuch as what proportion soever cones have, also cylinders, being triple of them, have the same; therefore cylinder to cylinder shall have proportion triple of the diameters of the bases.

PROP. XIII.

If a cylinder ABCD *be divided by a plane* EF *parallel to the opposite planes* BC, AD, *then as one cylinder* AEFD *is to the other cylinder.* EBCF, *so is the axis* GI *to the axis* IH.

The axis being produced,
a 3. 1. *a* take GK = GI, and HL = IH = LM. and conceive planes drawn at the points K, L, M, parallel to the circles AD, BC.
b 11. 12. *b* therefore the cylinder FD = the cyl. AN. and the cyl. EC *b* = BO *b* = OP. therefore the cylinder EN is as multiplex of the cylinder ED as the axis IK is of the axis IG. and in like manner the cylinder FP is as multiplex of the cylinder BF, as the axis IM is of the axis IH. but as IK is =, ⊏, ⊐
c 11. 12. IM, *c* so is the cylinder EN =, ⊏, ⊐ FP. *d* therefore
d 6. def. 5. the cylinder AEFD. the cyl. EBCF :: GI. IH. *W.W. to be Dem.*

PROP. XIV.

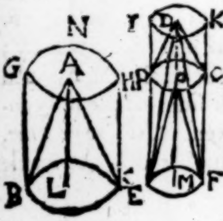
Cones AEB, CFD, *and cylinders* AH, CK, *consisting upon equall bases* AB, CD, *are to one another as their altitudes* ME, NF *are.*

The cylinder HA, and the axis EM being produced, take ML = FN; and at the point L draw a plane parallel to the
a 11. 12. base AB. *a* then shall the cyl. AP be = CK. *b* but the
b 13. 12. cyl. AH. AP (CK) :: ME. ML (NF.) *W.W. to be Dem.*

The same may be affirmed of cones subtriple of
* apply 9, and 7. 12. cylinders; * as also of prismes and pyramids.

PROP.

PROP. XV.



In equall cones BAC,EDF, *and cylinders* BH,EK, *the bases and altitudes are reciprocall* (BC. EF :: MD. LA:) *And cones and cylinders, whose bases and altitudes are reciprocall, are equall one to the other.*

If the altitudes be equall then the bases are equall too, and the thing is evident. If unequall, then take away MO = LA.

1. *Hyp.* Then is MD. MO (a LA) b :: the cyl. EK. (c BH) EQ d :: the cir. BC. EF. *Which was to be Dem.* a 14. 12. b constr. c hyp. d 11. 12.

2. *Hyp.* BC. EF e :: DM. OM (LA) f :: the cyl. EK. EQ g :: BC. EF h :: BH. EQ. k Therefore the cylind. EK = BH. *Which was to be Dem.* e hyp. f 11. 12. g 11. 5. h 11. 12. k 9. 5.

The same argument may be used for cones.

PROP. XVI.

Two unequall circles ABCG, DEF, *having the same centre* M, *to inscribe in the greater circle* ABCG *a polygonous figure of equall and even sides, which shall not touch the lesser circle* DEF.

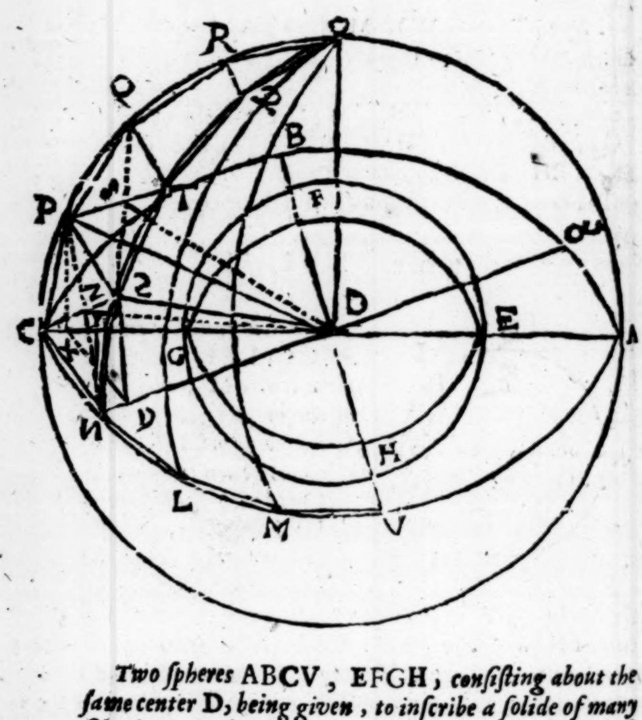
Through the center M draw the line AC cutting the circle DEF in F, from whence raise a perpendicular FH. a divide the semicircle ABC into two equall parts; and the half thereof BC also; and so do continually, b till the arch IC become lesse then the arch HC. from I let fall the perpendicular IL. It a 30. 3. b 1. 10.

 is

is manifest that the arch IC measures the whole circle, and that the number of arches is even, and so that the subtended line IC is the side [e] of the poly-
c *sch.* 16. 4. gonon that may be inscribed without touching the
d *cor.* 16. 3. lesser circle DEF. For HG [d] touches the circle DEF,
e 28. 1. *e* to which IK is parallel, and placed outwardly;
f 34. *def.* 1. *f* wherefore IK does not touch the circle DEF; much lesse do CI, CK, and the other sides of the polygonon more remote from the center. *W. W. to be Done.*

Coroll. Observe that IK touches not the circle DEF.

Prop. XVII.



Two spheres ABCV, EFGH, *consisting about the same center* D, *being given, to inscribe a solide of many sides (or Polyedron) in the greater sphere* ABCV, *which shall*

shall not touch the superficies of the lesser sphere EFGH.

Let both the spheres be cut by a plane passing by the center making the circles EFGH, ABCV; and the diameters AC, BV drawn, cutting perpendicularly. In the circle ABCV, a inscribe the equilaterall polygone VMLNC, &c. not touching the circle EFGH: then draw the diameter Nα, and erect DO perpendicular to the plane ABC. by DO, and by the diameters AC, Nα, conceive planes DOC, DON erected, which shall be b perpendicular to the circle ABCV, and so in the superficies of the sphere make c the quadrants DOC, DON. In which let the right lines CP, PQ, QR, RO, NS, ST, Tγ, γO d be fitted, equall, and of equall multitude with CN, NL, &c. make the same construction in the other quadrants OL, OM, &c. and in the whole sphere. Then I say the thing required is done.

a 16. 12.
b 18. 11.
c cor. 33. 6.
d 4 1.

From the points P, S, to the plane ABCV draw the perpendiculars PX, SY, e which shall fall on the sections AC Nα. Therefore because both f the right angles PXC, SYN, g and PCX, SNY insisting on h equall circumferences, f are equall, the triangles also PCX, SNY k are equiangular. Wherefore being PC k = SN, l also is PX = SY, l and XC = YN; m whence DX = DY. n and therefore DX. XC :: DY. YN. o therefore YX, NC are parallels. but because PX, SY are equall, and since being perpendicular to the same plane ABCV, they are also p parallels, q therefore YX, SP shall be equall and parallels. r whence SP, NC, are parallel one to the other; and so the s quadrilaterall NCPS, and by the same reason SPQT, TQRG, as also the t triangle γRO are so many planes. In like manner the whole sphere may be shewn full of such quadrilateralls and triangles. wherefore the figure inscribed is a polyedron.

e 38 11.
f 12. ax.
g 27. 3.
h 32. 1.
k constr.
l 26. 1.
m 3. ax. 1.
n 7. 5.
o 2. 6.
p 6. 11.
q 33. 1.
r 9. 11.
s 7. 11.
t 2. 11.

From the center D u draw DZ perpendicular to the plane NCPS; and join ZN, ZC, ZS, ZP. Because DN. NC x :: DY. YX. thence NC is y ⊏ YX

u 11. 11.
x 4 6.
y 14. 5.

(SP)

(SP,) and likewise SP $\sqsubset$ TQ, and TQ $\sqsubset$ γR. And because the angles DZC, DZN, DZS, DZP
z 3 def. 11. a 15. def. 11. b 47. 1. *z* are right, and the sides DC, DN, DS, DP, *a* equall, and DZ common, *b* thence ZC, ZN, ZS, ZP are equall one to the other; and consequently about the
c 15. def. 1. d constr. e 28. 3. quadrilaterall NCPS *c* a circle may be described, in which (because NS, NC, CP, are *d* equall, and NC $\sqsubset$ SP) NC *e* subtends more then the quadrant,
f 33. 9. g 12. 2. *f* therefore the ang. NZC at the center is obtuse. *g* therefore NCq $\sqsubset$ 2 ZCq (ZCq + ZNq.) Let NI be drawn perpendicular to AC. therefore since
h 32. 1. k 9. ax. 1. l 5. 1. the angle ADN (*h* DNC + DCN) *k* is obtuse, the half of it DCN shall be greater then the half of a right angle; and so that which remains of the right
m 19. 1. o 47. 1. p 47. 1. q cor. 19. 12. ang. CNI shall be lesse then it. *m* whence IN $\sqsubset$ IC. therefore NCq (NIq + ICq) *o* $\sqsupset$ 2 INq. therefore IN $\sqsubset$ ZC. and consequently DZ *p* $\sqsubset$ DI. but the point I is *q* without the sphere EFGH. & so much more the point Z. wherefore the plane NC-
r 47. 1. PS, (whose *r* next point to the center is Z) does not touch the sphere EFGH. And if a perpendicular Dδ be drawn to the plane SPQT, the point δ, & so also the plane SPQT is yet further removed from the center. which is also true of the other planes of the polyedron. Therefore the polyedron ORQPCN, &c. inscribed in the greater sphere, does not touch the lesser. *W.W. to be Done.*

Coroll.

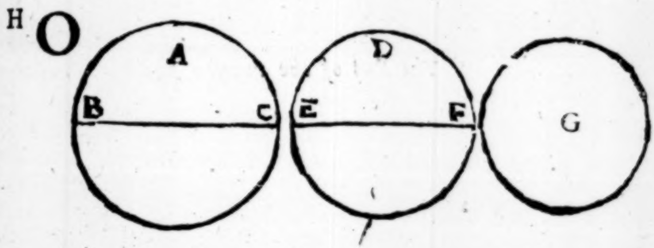
Hence it follows, *that if in any other sphere a solid polyedron, like to the abovesaid solid polyedron, be inscribed, the proportion of the polyedron in one sphere to the polyedron in the other is triple of that of the diameters of the spheres.*

For if right lines be drawn from the centers of the spheres to all the angles of the bases of the said polyedrons, then the polyedrons will be divided into pyramids equall in number and like; whose homo-

mologous ſides are ſemidiameters of the ſpheres; as appears, if the leſſer of theſe ſpheres be conceived deſcribed within the greater about the ſame center. For the right lines drawn from the center of the ſphere to the angles of the baſes will agree one to the other by reaſon of the likeneſſe of the baſes; and ſo will like pyramids be made. Wherefore ſince every pyramide in one ſphere to every pyramide like it in the other ſphere *a* ha's proportion triple to that[a] of the homologous ſides, that is, of the ſemidiameters of the ſpheres; and *b* as one pyramide is to one pyramide, ſo all the pyramids, that is, the ſolid polyedron compoſed of theſe, are to all the pyramids, that is, the ſolid polyedron compoſed of the others; thereſore the polyedron of one ſphere ſhall have to the polyedron of the other ſphere, proportion triple of that of the ſemidiameters, *c* and ſo of the diameters of the ſpheres.

a cor. 8. 12.
b 12. 5.
c 15. 5.

PROP. XVIII.



Spheres BAC, EDF, *are in triple proportion one to the other of that in which their diameters* BC, EF, *are.*

Let the ſphere BAC be to the ſphere G in triple proportion of that of the diameter BC to the diameter EF. I ſay $G = EDF$. For if it be poſſible, let G be $\sqsupset$ EDF. and conceive the ſphere G concentricall with EDF. In the ſphere EDF *a* inſcribe a polyedron not touching the ſphere G, and a like po-

a 17. 12.

polyedron in the ſphere BAC. Theſe polyedrons
b cor. 17.12. *b* are in triple proportion of the diameters BC, EF,
c hyp. *c* that is, of the ſphere BAC to G. *d* Conſequently
d 14.5. the ſphere G is greater then the polyedron inſcribed in the ſphere EDF, the part then the whole.

Again, if it be poſſible, let the ſphere G be ⊏ EDF. and as the ſphere EDF is to another ſphere
e hyp. inverſ. H, ſo let G be to BAC, *e* that is, in triple proportion
f 14.5. of the diameter EF to BC. therefore ſince BAC *f* ⊏ H, we ſhall incurre the abſurdity of the firſt part. wherefore rather the ſphere G = EDF. *W.W. to be Demonſtrated.*

Coroll.

Hence, As one ſphere is to another ſphere, ſo is a polyedron deſcribed in that to a like polyedron deſcribed in this.

The End of the twelfth Book.

THE

THE THIRTEENTH BOOK OF EUCLIDE'S ELEMENTS.

PROP. I.

If a right line z *be divided according to extream and mean proportion* (z.a :: a. e.) *the square of the half of the whole line* z, *and of the greater segment* a, *as one line, is quintuple to that which is described of half of that whole line* z.

I say $Q.\ a + \frac{1}{2}$
$z = 5\ Q: \frac{1}{2}\ z.$ a that a 4. 2.
is, $aa + \frac{1}{4} zz +$ b 3.ax.1.
c 2.2.
d hyp. and
$za = zz + \frac{1}{4} zz.$ b or $aa + za = zz.$ For $ze +$ 16.6.
e 2.ax.and
za c $= zz.$ and ze d $= aa.$ e therefore $aa + za =$ 1.ax.
$zz.$ *W.W. to be Dem.*

PROP. II.

See the 1. *Scheme.*

If a right line $\frac{1}{2} z + a$ *be in power quintuple to a segment of it self* $\frac{1}{2}$ z. *the line double of the said segment* (z) *being divided according to extreme and mean proportion, the greaer segment is* (a) *the other part of the right line at first given* $\frac{1}{2} z + a$.

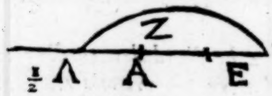
I say z.a :: a.e. For because by the hyp. * $aa +$ * 4. 2.
$\frac{1}{4} zz + za = zz + \frac{1}{4} zz$; or $aa + za = zz$ a $=$ a 2. 2.
b 3.ax.1
$ze + za$, b thence shall aa be $= ze.$ c wherefore z. a c 17. 6.
:: a. e. *W.W. to be Dem.*

PROP.

PROP. III.

If a right line z *be divided according to extreme and mean proportion* (z. a :: a.e.) *the line made of the lesse segment* e *and half of the greater segment* a, *is in power quintuple to the square, which is described of the half line of the greater segment* a.

I say Q:e + $\frac{1}{2}$ a = 5 Q:
a 4.2. $\frac{1}{2}$ a. *a* that is ee + $\frac{1}{4}$ aa +
b 3.ax. c 3.2. d hyp. and 17.6.

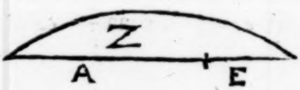


ea = aa + $\frac{1}{4}$ aa. *b* or ee + ea = aa. For ee + ea *c* = ze *d* = aa. *Which was to be Dem.*

PROP. IV.

If a right line z *be cut according to extreme and mean proportion* (z.a :: a.e) *the square made of the whole line* z, *&* *that made of the lesser segment* e, *both together, are triple of the square made of the greater segment* a.

a 4.1. I say zz + ee = 3 aa. *a* or aa + ee + 2 ae + ee = 3 aa. For
b 3.2. c 17.6. d 2.ax. ae + ee *b* = ze *c* = aa. *d* therefore aa + 2 ae + 2 ee = 3 aa. *W.W. to be Dem.*



PROP. V.

D A C B
———l———l———

If a right line AB *be cut according to extreme and mean proportion in* C, *and a line* AD, *equall to the greater segment* BC, *added to it, the whole right line* DB *is divided according to extreme and mean proportion; and the greater segment is the right line* AB *given at the beginning.*

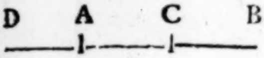
a hyp. For because AB.AD *a* :: AC. CB. and by inversion AD.AB :: CB. AC. therefore by composition DB.AB :: AB.AC (AD.) *W.W. to be Dem.*

Schol.

Schol.

But if BD. BA :: BA. AD. then shall be BA. AD :: AD. BA − AD. For by division is BD − BA (AD.) BA :: BA − AD. AD. therefore inversely BA. AD :: AD. BA − AD.

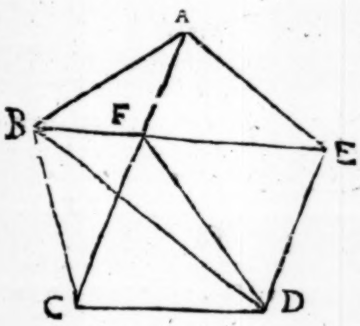
PROP. VI.



If a rationall right line AB *be cut according to extreme and mean proportion in* C, *either of the segments* (AC, CB) *is an irrationall line of that kind which is called apotome or residuall.*

To the greater segment AC *a* adde AD $=\frac{1}{2}$ AB. a 3. 1.
b therefore DCq $=$ 5 DAq. *c* therefore DCq^{2} $\sqsupset\!\sqsubset$ b 1. 13.
DAq. consequently *d* since AB, *e* and so the half there- c 6. 10.
of DA are $\dot{\rho}$, likewise DC is $\dot{\rho}$. But because 5. 1 :: d *hyp.*
not Q. Q. *f* thence is DC $\sqsupset\!\sqsubset$ DA. *g* therefore DC e *Sch.* 12. 10.
− AD, that is, AC, is a residuall line. Further, be- f 9. 10.
cause ACq *h* $=$ AB x BC, and AB is $\dot{\rho}$, *k* likewise g 74. 10.
BC is a residuall line. *W.W. to be Dem.* h 17. 6.
k 98. 10.

PROP. VII.



If three angles of an equilaterall Pentagone ABCDE; *whether they follow in order,* (EAB, ABC, BCD,) *or not,* (EAB, BCD, CDE) *be equall, the pentagone* ABCDE *shall be equiangular.*

Let

Let the right lines BE, AC, BD, be ſubtended to the equall angles in order.

a *hyp.* b 4. 1. c 4 *and* 5. 1. d 6. 1. e 3. *ax.* 1. Being the ſides EA, AB, BC, CD, and the included angles *a* are equall, *b* therefore ſhall the baſes BE, AC, BD, *c* & the angles AEB, ABE, BAC, BCA, be equall. *d* wherefore BF = FA, *e* and conſequently FC = FE; therefore the triangles FCD, FED, are equilaterall one to the other: f 8. 1. *f* whence the angle FCD = FED. g 2. *ax.* 1. *g* conſequently the ang. AED = BCD. In like manner the ang. CDE is equall to the reſt; wherefore the pentagone is equiangular. *W.W to be Dem.*

h 4. 1. k 5. 1. l 2. *ax.* But if the angles EAB, BCD, CDE, which are not in order, be ſuppoſed equall, *h* then ſhall the ang. AEB be = BDC, and BE = BD. *k* and thence the ang. BED = BDE. *l* conſequently the whole ang. AED = CDE. therefore becauſe the angles A, E, D, in order, are equall, as before, the pentagone ſhall be equiangular. *W.W. to be Dem.*

PROP. VIII.

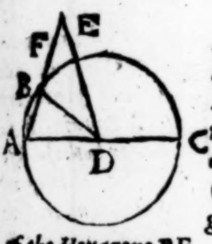
If in an equilaterall and equiangular Pentagone ABCDE, *two right lines* BD, CE, *ſubtend two angles* BCD, CDE, *following in order, thoſe lines do cut one another according to extreme and mean proportion; and their greater ſegments* BF *or* EF *are equall to the ſide of the Pentagone* BC.

a 14. 4. b 28. 3. c 27. 3. d 32. 1. e 33. 6. *a* Deſcribe about the pentagone the circle ABD. *b* The arch ED is = BC, *c* therefore the angle FCD = FDC. *d* therefore the ang. BFC = 2 FCD (FCD + FDC.) But the arch BAE *b* is = 2 ED, and conſequently the angle BCF *e* = 2 FCD = BFC. f 6. 1. *f* wherefore BF = BC. *Which was to be Dem.*

g 27. 3. h 4. 6. Moreover becauſe the triangles BCD, FCD, are *g* equiangular, *h* therefore BD. DC. (BF.) :: CD.

(BF.) FD. and likewise EC. EF :: EF. FC. *W.W. to be Dem.*

PROP. IX.



If the side of an Hexagone BE, *and the side of a Decagone* AB, *both described in the same circle* ABC, *be added together, the whole right line* AE *is cut according to extreme and mean proportion* (AE. BE :: BE. AB,) *and the greater segment thereof is the side of the Hexagone* BE.

Draw the diameter ADC, and join the right lines DB, DE. Because the ang. BDC *a* = 4 BDA, and the ang. BDC *b* = 2 DBA (DAB + DBA) thence shall DBA (*b* BDE + BED) *c* be = 2 BDA *d* = 2 BDE. whence the ang. DBA or DAB *e* = ADE. Therefore the triangles ADE, ADB, are equiangular: *f* wherefore AE. AD (*g* BE) :: AD. (BE.) AB. *W.W. to be Dem.*

a hyp. and 27 3.
b 32. 1.
c 7 ax. 1.
d 5. 1.
e 1 ax 1.
f 4. 6.
g cor. 15. 4.

Coroll.

Hence, If the side of a hexagone in a circle be cut according to extreme and mean proportion; the greater segment thereof shall be the side of the Decagone in the same circle.

PROP. X.

If an equilaterall Pentagone ABCDE *be described in a circle* ABCE, *the side of the pentagone* AB *containeth in power both the side of a hexagone* FB, *and the side of a decagone* AH *described in the same circle.*

Draw the diameter AG. and bisect equally the arch AH in K. and draw FK, FH, FB, BH, HM.

a 28.3. and 3. ax.
b hyp. and 7. ax.
c 33. 6.
d 20. 3.
e 1. ax. 1.
f 32. 1.
g 4 6
h 17. 6
k 27 3.
m 4. 1.

The semicircle AG $-$ the arch AC a $=$ AG $-$ AD. that is, the arch CG $=$ GD b $=$ AH $=$ HB. therefore the arch BCG $=$ 2 BHK; c and so the ang. BFG $=$ 2 BFK. d but the ang. BFG $=$ 2 BAG. e therefore the ang. BFK $=$ BAG. Wherefore the triangles BFM, FAB, f are equiangular. g whence AB. BF :: BF. BM. h therefore AB x BM $=$ BFq. Moreover, the ang. AFK k $=$ HFK, and FA $=$ FH. m wherefore AL $=$ LH, m and the angles FLA, FLH are equall, and so right angles. therefore the ang. LHM m $=$ LAM n $=$ HBA. therefore the triangles AHB, AMH, o are equiangular. wherefore AB. AH :: AH. AM. q therefore AB x AM $=$ AHq. So that seeing ABq r $=$ AB x BM $+$ AB x AM, s thence ABq $=$ BFq $+$ AHq. *W.W. to be Dem.*

n 27. 3.
o 32. 1.
p 4. 6.
q 17. 6.
r 2. 2.
s 2. ax.

Coroll.

1. Hence, a right line (FK) which being drawn from the center (F) divides an arch (HA) into two equall segments, do's also divide the right line (HA)

sub-

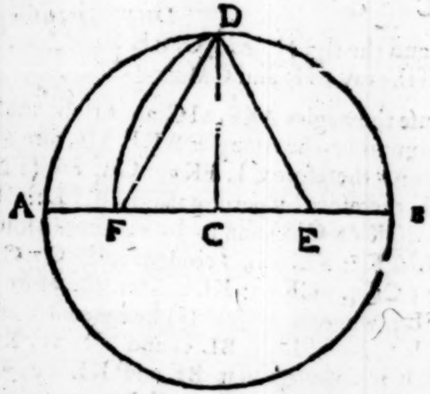
subtending that arch, perpendicularly into two equall segments,

2. The diameter of a circle (AG) drawn from any angle (A) of a pentagone, do's divide equally in two both the arch (CD) which the side of the pentagone opposite to that angle subtends, and also the opposite side it self (CD) and that perpendicularly.

Schol.

Here, according to our promise, we shall lay down a ready praxis of the 11. prop. *of the* 4. *Book.*

Probl.



To find out the side of a pentagone to be inscribed in a circle ADB.

Draw the diameter AB, to which erect a perpendicular CD at the center C, divide CB equally in E. and make EF = ED. then DF shall be the side of the pentagone.

For $BF \times FC + ECq$ [a] $= EFq$ [b] $= EDq$ [c] $=$ DCq + ECq. [d] therefore BF x FC = DCq or BCq. [e] wherefore BF. BC :: BC. FC. therefore since BC is the side of a hexagone, [f] FC shall be the side of a decagone. Consequently DF [h] $= \sqrt{DCq + FCq}$ [g] is the side of a pentagone. *W.W. to be Done.*

a 6. 2.
b constr.
c 47. 1.
d 3. ax.
e 17. 6.
f 9. 13.
g 10. 13.
h 47. 1.

PROP. XI.

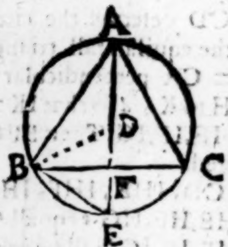
If in a circle ABCD, *whose diameter is rationall* AG, *an equilaterall pentagone be inscribed* ABCDE; *the side of the pentagone* AB *is an irrationall line, of that kind which is called a minor line.*

Draw the diameter
* 10. 6. BFH, and the right lines AC, AH; and * make FL $= \frac{1}{4}$ of the ray FH; and CM $= \frac{1}{4}$ CA.

a cor. 10. 13. Because the angles AKF, AIC, are *a* right angles, &
b 32. 1. CAI commune, the triangles AKF, AIC, are *b* equi-
c 4. 6. angular: *c* therefore CI. FK *c* :: CA. FA (FB) *d* ::
d 15. 5. CM.FL. therefore by permutation FK.FL :: CI.CM
e 18. 5. *d* :: CD.CK (2 CM) and so by *e* composition CD
f 22. 6. + CK. CK :: KL. FL. *f* consequently Q: CD +
g 1. 13. CK (*g* 5 CKq.) CKq :: KLq. FLq. therefore KLq
= 5 FLq. wherefore if BH ($\dot{\rho}$) be taken 8, FH shall
be 4, FL 1, and FLq 1, BL 5, and BLq 25, KLq 5.
h 9. 10. by which it appears that BL and KL are $\dot{\rho}$ *h* ⊤,
k 74. 10. *k* and so BK is a residuall, and KL its congruent
l 9. 10. or adjoining line. but being BLq − KLq = 20,
m cor 8. 6. *l* thence BL ⊓ √BLq − KLq. *m* whence BK shall
and 17. 6. be a fourth residuall line. Therefore because ABq
n 95. 10. *m* is = HB x BK, *n* shall AB be a minor line. *Which was to be Dem.*

PROP.

PROP. XII.



If in a circle ABEC *an equilaterall triangle* ABC *be inscribed, the side of that triangle* AB *is in power triple to the line* AD *drawn from* D *the centre of the circle to the circumference.*

The diameter being extended to E, draw BE. Because the arch BE *a* = EC, the arch BE is the sixt part of the circumference. *b* therefore BE = DE. hence AEq *c* = 4 DEq (4 BEq) *d* = ABq + BEq (+ ADq.) *e* consequently ABq = 3 ADq. *W.W. to be Dem.*

a cor. 10, 13.
b cor. 15. 4.
c 4. 2.
d 47. 1.
e 3. ax. 1.

Coroll.

1. AEq. ABq :: 4. 3.

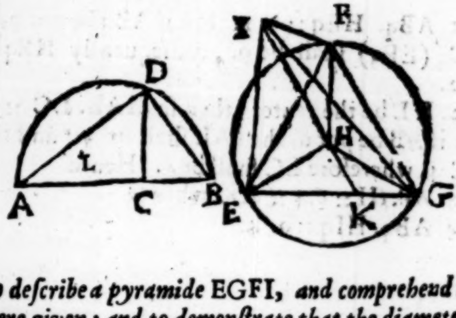
2. ABq. AFq :: 4. 3. *f* For ABq. AFq :: AEq. ABq.

3. DF = FE. For the triangle EBD *g* is equilaterall, *h* and BF perpendicular to ED. *h* therefore EF = FD.

4. Hence, AF = DE + DF = 3 DF.

f cor. 8. 6. and 22. 6.
g cor. 15. 4.
h cor. 3. 3.

PROP. XIII.



To describe a pyramide EGFI, *and comprehend it in a sphere given: and to demonstrate that the diameter of the sphere* AB *is in power sesquialtera of the side* EF *of the pyramide* EGFI.

a 10. 6. About AB describe the semicircle ADB; a and let AC be = 2 CB. from the point C erect the perpendicular line CD; and join AD, DB. then at the intervall of the ray HE = CD describe the circle b cor. 15. 4. HEFG. b wherein inscribe the equilaterall triangle c 12. 11. d 3. 1. EFG. from H c erect IH = CA perpendicular to the plane EFG. produce IH to K, d so that IK = AB; and join the right lines IE, IF, IG. Then EFGI shall be the pyramide required.

For because the angles ACD, IHE, IHF, IHG, e constr. e are right angles; & CD, HE, HF, HG e equall, e & f 41. 1. g 20. 6. IH = AC; f therefore AD, IE, IF, IG shall be equall among themselves. But being AC (2 CB.) CB g :: h 2. ax. ACq. CDq. thence shall ACq be = 2 CDq. therek 12. 13. l 1. ax. 1. fore ADq f = ACq + CDq h = 3 CDq = 3 HEq k = EFq. l therefore AD, EF, IE, IF, IG are equall, and so the pyramide EFGI is equilateral. But if the point C be placed upon H, and AC upon HI, the m 8. ax. right lines AB, IH, m shall agree, as being equall. Wherefore the semicircle ADB being drawn about n 15. def. 1. * 31. def. 11. the axis AB or IK n shall passe by the points E, F, G, * and so the pyramide EFGI shall be inscribed in a sphere. *W. W. to be Done.*

o cor. 8. 6. p constr. Also it is manifest that BAq. ADq o :: BA. AC p :: 3. 2. *W. W. to be Dem,*

Coroll.

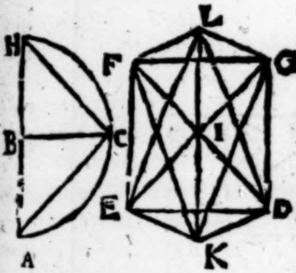
1. ABq. HEq :: 9. 2. For if ABq be put 9, then q 12. 13. ACq (EFq) shall be 9. q consequently HEq shall be 2.

2. If L be the center, then shall AB. LC :: 6. 1. r constr. For if AB be put 6, then AL shall be 3. r and thence AC 4, wherefore LC shall be 1. Hence

3. AB. HI :: 6. 4 :: 3. 2. whence

4. ABq. HIq :: 9. 4.

PROP. XIV.



To describe an Octaedron KEFGDL, *and comprehend it in the given sphere, wherein a pyramide is: and to demonstrate that* AH *the diameter of the sphere is in power double of* AC *the side of that Octaedron.*

About AH describe the semicircle ACH. and from the center B erect the perpendicular BC. draw AC, HC. then upon ED = AC *a* make the square EFGD, whose diameters DF, EG, cut in the center I. from I draw IL = AB *b* perpendicular to the plane EFGD. produce IL, *c* till IK = IL. and join KE, KF, KG, KD, LE, LF, LG, LD; then shall KEFGDL be the Octaedron required. a 46. 1. b 12. 11. c 3. 1.

For AB, BH, FI, IE, &c. being semidiameters of equall squares are equall one to the other. *d* whence the bases LF, LE, FE, &c. of the rightangled triangles LIE, LIF, FIE, &c. are equall, and consequently the eight triangles LFE, LFG, LGD, LDE, KEF, KFG, KGD, KDE, are equilaterall, *e* and make an Octaedron, which may be inscribed in a sphere, whose center is I, and IL or AB the radius. (because AB, IL, IF, IK, &c. *f* are equall.) *W. W. to be Done.* Moreover, it is evident that AHq (LKq) *g* = 2 ACq (2 LDq.) *W. W. to be Dem.* d 4. 1. e 27. def. 11. f constr. g 47. 1.

Coroll.

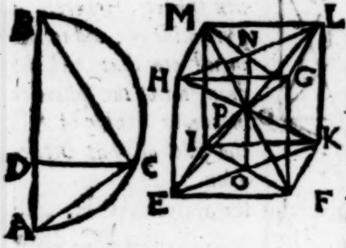
1. Hence it is manifest, that in the octaedron the three diameters EG, FD, LK doe cut one the other perpendicularly in the center of the sphere.

2. Also that the three planes EFGD, LEKG, LFKD are squares, cutting one another perpendicularly.

3. The Octaedron is divided into two like and equall pyramids EFGDL, and EFGDK, whose common base is the square EFGD.

15. 11. 4. Lastly, it follows that the opposite bases of the octaedron are parallel one to the other.

PROP. XV.



To describe a cube EFGHIKLM, *and comprehend it in the same sphere wherein the former figures were; and to demonstrate that* AB *the diameter of the sphere is in power triple to* EF *the side of that cube.*

a 10. 6. Upon AB describe a semicircle ACB; a and make
b 46. 11. AB = 3 DA. from D raise the perpendicular DC, & join BC and AC. Then upon EF = AC b make the square EFGH, upon whose plane let the right lines EI, FK, HM, GL, stand perpendicular, being equall to EF, & connect them with the right lines IK, KL, LM, IM. The solid EFGHIKLM is a cube, as is sufficiently apparent from the construction.

In the opposite squares EFKI, HGLM, draw the diameters EK, FI, HL, MG, by which let the planes EKLH, FIMG be drawn, cutting one another in the
c cor. 39 11. line NO. which c shall divide equally in two parts
d 15 def. 1. and 14 def. 11. the diameters of the cube EL, FM, GI, HK, in P the center of the cube. d therefore P shall be the center of a sphere passing by the angular points of the cube.
e 47. 1
f constr
g cor. 8. 6.
h 14. 5.
Moreover, ELq e = EKq + KLq e = 3 KLq, f or 3 ACq. but ABq. ACq g :: BA. DA f :: 3. 1. h therefore AB = EL wherefore we have made a cube, &c. *W. W. to be Done.*

Coroll.

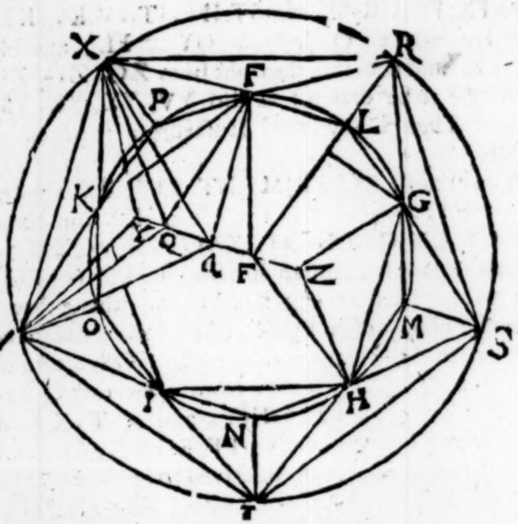
1. Hence it is manifest, that all the diameters of the cube are equall one to another, and do equally bisect one another in the center of the sphere. And by the same means the right lines which conjoin the

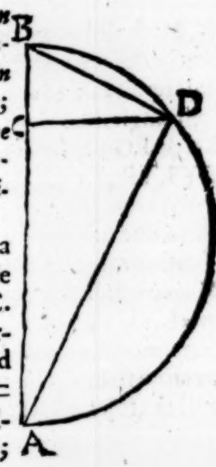
cen-

centers of the opposite squares are equally bisected in the same center.

2. The diameter of a sphere containeth in power the side of a tetraedron and of a cube, *viz.* ABq k = l BCq + m ACq. k 47. 1. l 13, 15. m 15. 13.

Prop. XVI.





To describe an Icosaedron ZGHIKFYVXRST, *and encompasse it in the sphere, wherein were contained the foresaid solids; and to demonstrate that* FG *the side of the Isocaedron is that irrationall line, which is called a minor line.*

Upon AB the diameter of a sphere describe the semicircle ADB; & a make AB = 5 BC. then from C erect CD perpendicular, and draw AD and BD. At the distance EF = BD describe the circle EFKNG; a 10. 6.

b 11.4 NG; *b* wherein inscribe the equilaterall pentagone FKIHG. Divide equally in two parts the arches FG, GH, &c. and join the right lines FL, LG, &c. being the sides of a decagone. Then *c* erect EQ, LR, MS,
c 12.11. NT, OV, PX equall to EF, and perpendicular to the plane FKNG; and connect RS, ST, TV, VX, XR; as also FX, FR, GR, GS, HS, ST, HT, IT, IV, KV, KX. Lastly, produce EQ, and take QY = FL, and EZ = FL. and conceive the right lines ZG, ZH, ZI, ZK, ZF to be drawn; as also YV, YX, YR, YS, YT. Then I say the Icosaedron required is made.

d constr. e 6 11. f 33. 1. For because EQ, LR, MS, NT, OV, PX, are *d* equall and *e* parallel, also those lines that join them EL, QR, EM, QS, EN, QT, EO, QV, EP, QX, *f* are equall & parallel. And thence likewise LM (or FG)
g 15.11. RS, MN, ST, &c. are equall one to the other. *g* therefore the plane drawn by EL, EM, &c. is distant equally from the plane passing by QR, QS, &c. *h* and
h 1. def. 3. the circle QXRSTV drawn from the center Q is equall to the circle EPLMNO; and RSTVX is an equilaterall pentagone. But EF, EG, EH, &c. and QX, QR, QS &c. being conceived to be drawn; then
k 47. 1. l constr. m 10. 13. n sch 48. 1. and 1 ax. because FRq *k* = FLq + LRq, *l* or EFq *m* = FGq, *n* therefore FR, FG, and so all RS, FG, FR, RG, GS, GH, &c. shall be equall one to the other. and consequently the ten triangles RFX, RFG, RGS, &c. are equilaterall and equall. Moreover, because XQY is a
o cor 14. 11. p 47. 1. q 10. 13. *o* right angle, therefore XYq *p* = QXq + QYq *q* = VXq or FGq. wherefore XY, VX, and likewise YV, YT, YS, YR, ZG, ZH, &c. are equall. Therefore other ten triangles are made, equilaterall and equall both to one another, and to the ten former; and so an Icosaedron is made.

Moreover, divide equally EQ in α, draw the right
r 15. def. 1. lines αF, αX, αV; and because QX *r* = QV, and αQ the common side, and FQX, EQV are right angles,
s 4. 1. *s* therefore shall αX be = αV; and by the same reason all the lines αX, αR, αS, αT, αV, αF, αG, αH, αI, αK are

are equall. But because ZQ.QE [t] :: QE. ZE. therefore Zαq [u] = 5 Eαq [x] = EQq (EFq) + Eαq [y] = αFq. therefore Zα = αF. [z] in like manner αF = Yα. therefore the sphere, whose center is α, and αF the ray, shall passe by the 12 angular points of the Icosaedron. t 9. 13. u 3. 13. x 4. 2. y 47. 1. z 15. 5.

Lastly, because Zα. αE :: ZY. QE; [a] and so Zαq. αEq :: ZYq. QEq. [b] therefore ZYq = 5 QEq, or 5 BDq = : but ABq. BDq [c] :: AB. BC :: 5. 1. [d] therefore ZY = AB. *W.W. to be Done.* a 22. 6. b 14. 5. c cor. 8. 6. d 1. ax. 1. e sch. 12. 10. f 11. 13.

Therefore if AB be put $\dot{\rho}$, [e] then EF = √ ABq shall be also $\dot{\rho}$. and consequently FG the side of the pentagone, and likewise of the Icosaedron, [f] is a minor line. *Which was to be Demonstrated.*

Coroll.

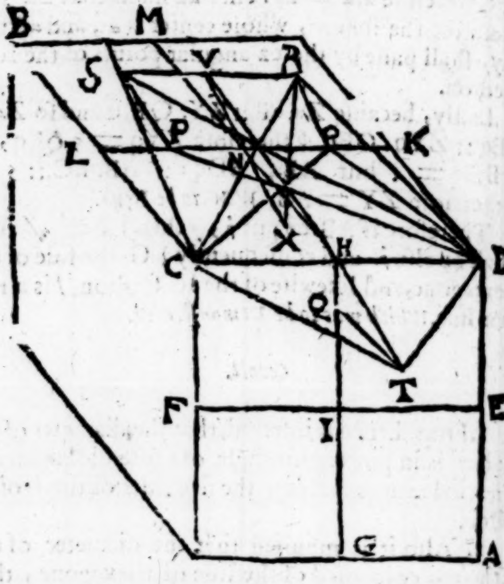
1. From hence is inferred, that the diameter of the sphere is in power quintuple of the semidiameter of the circle encompassing the five sides of the Icosaedron.

2. Also it is manifest that the diameter of the sphere is composed of the side of a hexagone, that is, of the semidiameter, and two sides of the decagone of a circle encompassing the five sides of the Icosaedron.

3. It appears likewise that the opposite sides of an Icosaedron, such as RX, HI, are parallels. For RX [a] is parall. to LP. [b] parall. to HI. a 33. 1. b sch. 16. 3.

PROP.

PROP. XVII.



To describe a Dodecaedron, and comprehend it in the sphere wherein the former figures were comprehended: and to demonstrate that the side RS *of the Dodecaedron is an irrationall line of that sort which is called an apotome or residuall line.*

Let AB be a cube inscribed in the given sphere, and let all the sides thereof be divided equally in the points E, H, F, G, K, L, &c. and join the right
a 30.6. lines KL, MH, HG, EF. a make HI.IQ :: IQ.QH; and take NO, NP, = IQ. then erect OR, PS, perpendicular to the plane DB, and QT to the plane AC; and let OR, PS, QT, be equall to IQ, NO, NP. whence DR, RS, SC, CT, DT, being connected, DRSCT shall be a pentagone of the dodecaedron required. For draw NV parallel to OR, and having drawn NV out as farre as the center of the cube X, join

join the right lines DS, DO, DP, CR, CP, HV, HT, RX. Because DOq *a* = DKq (*b* KNq) + KOq *c* = 3 ONq (3 ORq) *d* thence DRq = 4 ORq *e* = OPq, or RSq. therefore DR = RS. By the same reason DR, RS, SC, CT, TP are equall. But because OR *f* is = and *g* parallel to PS, therefore RS, OP, and *h* consequently RS, DC shall be also parallels. *h* therefore these with them that conjoin them DK, CS, VH, are in one and the same plane. Moreover, because HI. IQ *k* :: IQ. (TQ.) QH *k* :: HN. NV. and both TQ, HN, and QH, NV *k* are perpendicular to the same plane, *l* and so likewise parallels, *m* THV shall be a right line. *n* therefore the Trapezium DRSC, and the triangle DTS are in one plane extended by the right lines DC, TV. *o* therefore DCTSR is a pentagone, and that also equilaterall, by what is shewn already. Furthermore, because PK. KN :: KN. NP; and DSq *p* = DPq + PSq (PNq) = *p* DKq + PKq + NPq, *q* thence DSq = DKq + 3 KNq = 4 DKq (4 DHq) *r* = DCq. therefore DS = DC. whence the triangles DRS, DCT, are equilaterall one to another. *s* therefore the angle DRS = DTC, and likewise the ang. CSR = DCT. therefore the pentagone DTCSR is also equiangular. Moreover, because AX, DX, CX, &c. are semidiameters of the cube, *t* thence is XN = IH or KN, *u* and so XV = KP; wherefore because RVX, is a *x* right angle, *z* thence RXq = XVq + RVq (NPq) = KPq + NPq *a* = 3 KNq *b* = AXq or DXq, &c. therefore RX, AX, DX, and by the same reason XS, XT, AX, are equall one to another. And if by the same method, whereby the pentagone DTCSR was made, twelve like pentagones, touching the twelve sides of the cube, be made, they shall compose a Dodecaedron; and a sphere passing by their angular points, whose ray is AX or RX, shall comprehend that Dodecaedron. *W. W. to be Done*.

a 47. 1. b 7. ax. 1. c 4. 13. d 47. 1. e 4. 2. f constr. 96. 11. g 33. 1. h 9. 1. k 7. 11. l 6. 11. m 32. 6. n 1. & 2. 11. o 5. 13. p 47. 1. q 1. ax. 2. & 4. 13. r 4. 2. s 8. 1. t 15. 13. u 1. ax. 1. x 29. 1. z 47. 1. a 4. 13. b 15. 13.

Lastly, because KN. NO *c* :: NO. OK. *d* thence KL.

c constr. d 15. 5.

KL.OP :: OP.OK + PL. Therefore if AB the diameter of the sphere be supposed ṗ, then shall KL e = √ $\frac{AB}{3}$ f be also ṗ. g whence OP or RS the side of the dodecaedron shall be a residuall line. *W.W. to be Dem.*

e 15. 13.
f sch. 12. 10
g 6. 13.

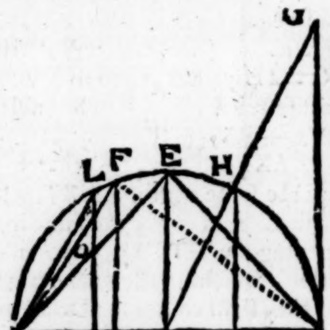
Coroll.

From this demonstration it follows, 1. that if the side of a cube be cut in extreme and mean proportion, the greater segment shall be the side of the dodecaedron described in the same sphere.

2. If the lesser segment of a right line, cut in extreme and mean proportion, be the side of the dodecaedron, the greater segment shall be the side of the cube inscribed in the same sphere.

3. It is manifest also, that the side of the cube is equall to the right line which subtendeth the angle of a pentagone of the dodecaedron inscribed in the same sphere.

PROP. XVIII.



To find out the sides of the precedent five figures, and compare them together.

Let AB be the diameter of the sphere given, and AEB the semicircle, and let AC be a = $\frac{1}{2}$ AB, and AD b = $\frac{1}{3}$ AB. then erect the perpendiculars CE, DF, and BG = AB. join AF, AE, BE, BF, CG; and let fall the perpendicular HI from H;

a 10. 1.
b 10. 6.

and

and CK being taken equall to CI, from K erect the perpendicular KL, and join AL. Laſtly, c make AF. AO :: AO. OF. c 30. 6.

Therefore 3. 2 d :: AB. BD e :: ABq. BFq the ſide of a Tetraedron. and 2. 1 :: a AB. AC :: ABq. BBq f the ſide of an Octaedron. d *conſtr.* e *cor.* 8. 6. f 14. 13.

Alſo 3. 1 d :: AB. AD e :: ABq. AFq, g the ſide of an Hexaedron. g 15. 13. h *conſtr.*

Moreover, becauſe AF. AO h :: AO. OF. k thence ſhall AO be the ſide of a Dodecaedron. Laſtly, BG, (2 BC.) BC l :: HI. IC. m therefore HI = 2 CI n = KI. therefore HIq o = 4 CIq. conſequently CHq p = 5 CIq. q therefore ABq = 5 KIq. r therefore KI or HI is a ray of a circle encloſing the pentagone of an Icoſaedron; & AK or IB r is the ſide of a decagone inſcribed in the ſame circle. ſ whence AL ſhall be the ſide of a pentagone, t and alſo the ſide of an Icoſaedron. Whereby it appears that BF, BE, AE are ṗ ⊐. and AL, AO ṗ ⊐, and BF ⊏ BE, and BE ⊏ AF, and AF ⊏ AO. And becauſe 3 AFq = ABq u = 5 KLq, and AF x AO ⊏ AF x OF, x and ſo AF x AO + AF x OF ⊏ 2 AF x OF, y that is, AFq ⊏ z 2 AOq. a thence ſhall 3 AFq (5 KLq) be ⊏ 6 AOq. conſequently KL ⊏ AO; and much rather AL ⊏ AO. k *cor* 17. 13. l 4. 6. m 14. 5. n *conſtr.* o 4. 2. p 47. 1. q 15. 5. r *cor.* 16. 13. ſ 10. 13. t 16. 13. u 1. 6. x 4 *ax.* 1. y 1. 2. z 17. 6. a 47. 1.

That we may expreſſe theſe ſides in numbers; If AB be ſuppoſed √60, then, reducing what is already ſhewn to ſupputation, BF = √40, & BE = √30, & AF = √20. Alſo AL = √: 30 − √180 (for AK = √15 − √3. and KL (HI) = √12:) Laſtly AO = √: 30 − √500 (√25 − √5.)

Schol.

Schol.

*It is very apparent that besides the five aforesaid figures, there cannot be described any other regular solid figure (*viz.*such as may be contained under ordinate and equall plane figures.)*

For three plane angles at least are required to the
a 21.11. constituting of a solid angle ; *a* all which must be
b See schol. 32. 1. lesse then four right angles. *b* But 6 angles of an equilaterall triangle,4 of a square , and six of a hexagon, do severally equall 4 right angles;& 4 of a pentagon,3 of a heptagon, 3 of an octagone, &c. do exceed 4 right angles : Therefore only of 3, 4, or 5 equilaterall triangles,of 3 squares,or 3 pentagones, it is possible to make a solid angle. Wherefore besides the five above mentioned, there cannot be any other regular bodies.

Out of P. Herigon.

The Proportions of the sphere and the five regular figures inscribed in the same.

Let the diameter of the sphere be 2.then shall

The Peripherie or circumference of the greater circle, be 6 28318.

The superficies of the greater circle,3 14159.

The superficies of the sphere,12 56637.

The solidity of the sphere,4 1879.

The side of the tetraedron, 4 62299.

The

The ſuperficies of the tetraedron, 4 |6188.

The ſolidity of the tetraedron, 0 |15132.

The ſide of the Hexaedron, 1 |1547.

The ſuperficies of the hexaedron, 8.

The ſolidity of the hexaedron, 1 |5396.

The ſide of the Octaedron, 1 |41421.

The ſuperficies of the octaedron, 6 |9282.

The ſolidity of the octaedron, 1 |33333.

The ſide of the Dodecaedron, 0 |71364.

The ſuperficies of the dodecaedron, 10 |51462.

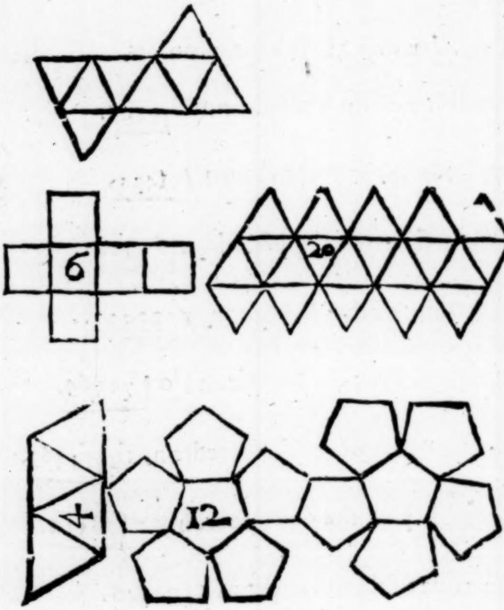
The ſolidity of the dodecaedron, 2 |78516.

The ſide of the Icoſaedron, 1 |05146.

The ſuperficies of the Icoſaedron, 9 |57454.

The ſolidity of the Icoſaedron, 2 |53615.

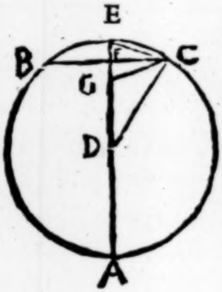
If five equilaterall and equiangular figures, like these in the schemes beneath, be made of paper, and rightly folded, they will represent the five regular bodies.



The End of the thirteenth Book.

THE FOURTEENTH BOOK OF EUCLIDE'S ELEMENTS.

Prop. I.

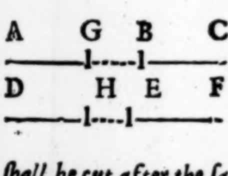


A Perpendicular line DF *drawn from* D *the center of a circle* ABC *to* BC *the side of a pentagone inscribed in the said circle, is the half of these two lines taken together,* viz. *of the side of the hexagone* DE, *and the side of the decagone* EC *inscribed in the same circle* ABC.

Take FG = FE, and draw CG: a Then CE is = CG. therefore the ang. CGE b = CEG b = ECD. therefore the ang. ECG c = EDC d = $\frac{1}{4}$ ADC e = $\frac{1}{2}$ CED ($\frac{1}{2}$ ECD.) f consequently the angle GCD = ECG = EDC. g wherefore DG = GC (CE.) therefore DF = CE (DG) + EF = $\frac{DE + CE}{2}$. *W.W. to be Dem.*

a 4. 1.
b 5. 1.
c 32. 1.
d hyp and 33. 6.
e 20. 3.
f 7. ax.
g 6. 1.

Prop. II.



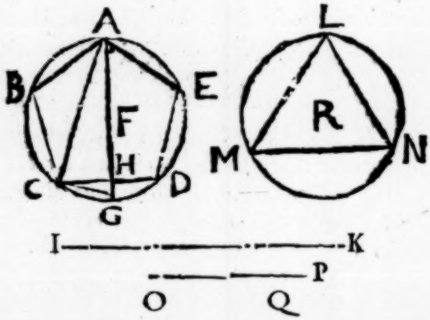
If two right lines AB, DE, *be cut according to extreme and mean proportion* (AB. AG :: AG. GB. *and* DE. DH :: DH. HE.) *they shall be cut after the same manner,* viz. *into the same proportions* (AG. GB :: DH, HE.)

Take

a 17. 6. b 8. 2. c 1. ax. 1. d 22. 5. and 22. 6. e 22. 5. f 17. 5.

Take BC = BG; and EF = EH. Then AB x BG is *a* = 5 AGq. wherefore ACq *b* = 4 ABG + AGq *c* = 5 AGq. In like manner shall DFq be = 5 DHq. *d* therefore AC. AG :: DF. DH. whence by addition AC + AG. AG :: DF + DH. DH. that is, 2 AB. AG :: 2 DE. DH *e* consequently AB. AG :: DE. DH; *f* whence by division AG. GB :: DH. HE. *W.W. to be Dem.*

PROP. III.

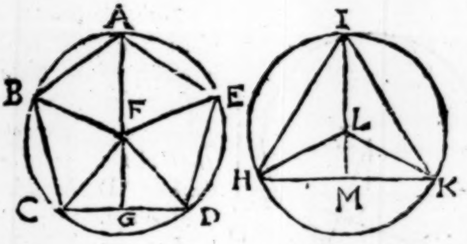


The same circle ABD *comprehends both* ABCDE *the pentagone of a Dodecaedron, and* LMN *the triangle of an Icosaedron inscribed in the same sphere.*

a sch. 47. 1. b 30. 6. c 47. 1. d 4. 2. e 10. 13. f 1, & 3. ax. g 8. 13. h 2. 13. & 16. 5. k 22. 6. and 4. 5. l 15. 13. m constr. n cor. 16. 13. o 12. 13. p 10. 13. q 15. 5. * before. r 1. ax. 1. and sch. 48. 1. s 1. def. 3.

Draw the diameter AG, and the right lines AC, CG. and let IK be the diameter of the sphere, *a* and IKq = 5 OPq. *b* & make OP. OQ :: OQ. QP. Because ACq + CGq *c* = AGq *d* = 4 FGq; & ABq *e* = FGq = CGq. *f* thence ACq + ABq = 5 FGq. moreover, because CA. AB *g* :: AB. CA − AB; and OP. OQ :: OQ. QP. *h* and so CA. OP :: AB. OQ. *k* therefore 3 ACq (*l* IKq.) 5 OPq (*m* IKq) :: 3 ABq. 5 OQq. therefore 3 ABq = 5 OQq. But because ML *n* is the side of a pentagon inscribed in the circle, whose ray is OP, thence 15 RMq *o* = 5 MLq *p* = 5 OPq + 5 OQq = * 3 ACq + 3 ABq *q* = 15 FGq. *r* therefore RM = FG. *s* and consequently the circle ABD is = to the circle LMN. *W.W. to be Dem.*

PROP. IV.



If from F *the center of a circle encompassing the pentagone of a dodecaedron* ABCDE, *a perpendicular line* FG *be drawn to one side of the pentagone* CD; *the rectangle contained under the said side* CD *and the perpendicular* FG, *being thirty times taken, is equall to the superficies of the Dodecaedron. Also,*

If from the center L *of a circle inclosing the triangle of an Icosaedron* HIK, *a perpendicular line* LM *be drawn to one side of the triangle* HK, *the rectangle contained under the said side* HK, *&* *the perpendicular* LM, *being thirty times taken, shall be equall to the superficies of the Icosaedron.*

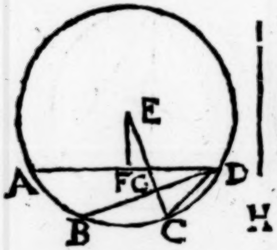
Draw FA, FB, FC, FD, FE. *a* then shall the triangles CFD, DFE, EFA, AFB BFC be equall. but CD x FG *b* = 2 triangles CFD. therefore 30 CD x GF *c* = 60 CFD *d* = 12 pentagones ABCDE *e* = to the superficies of the dodecaedron. *W.W. to be Dem.* a 8. 1. b 4. 1. c 15. 5. d 6 ax. e 17. 3.

Draw LI, LH, LK; then HK x LM *f* is = 2 triang. LHK. therefore 30 HK x LM *g* = 60 HLK = 20 HIK *h* = to the superficies of the Icosaedron. *W.W. to be Dem.* f 41. 1. g 15. 5. h 16. 13.

Coroll.

CD x FG. HK x LM *k* :: the superficies of the dodecaedron to the superf. of the Icosaedron. k 15. 5.

Prop. V.



The superficies of a Dodecaedron hath to the superficies of an Icosaedron inscribed in the same sphere, the same proportion that H *the side of a cube hath to* AD *the side of an Icosaedron.*

Let the circle ABCD
a 3. 14. a enclose both the pentagone of a dodecaedron, and the triangle of an Icosaedron; whose sides are BD, AD. upon which from the center E let fall the perpendiculars EF, EGC; and draw CD.

b 9. 13. Because EC + CD. EC b :: EC. CD. thence EG.
c 1. 14. d cor. 12. 13. ($c\ \frac{1}{2}$ EC + CD.) EF ($d\ \frac{1}{2}$ EC) e :: EF. EG − EF
e 15. 5. f cor. 17. 13. ($\frac{1}{2}$ CD.) but H. BD f :: BD. H − BD. g therefore
g 2. 14. h 1. 6. H. BD :: EG. EF. consequently H x EF = BD x
k 7. 5. EG. wherefore since H. AD h :: H x EF. AD x EF.
l cor. 4. 14. k thence shall be H. AD :: BD x EG. AD x EF l :: the superficies of a dodecaedron to the superficies of an Icosaedron. *W. W. to be Dem.*

Prop. VI.

If a right line AB be cut in extreme and mean proportion, then as the right line BF, containing in power that which is made of the whole line AB, and that which is made of the greater segment AC, is to the right line E containing in power that which is made of the whole line AB, and that which is made of the lesser segment BC; so is the side of the cube BG to the side of an Icosaedron BK inscribed in the same sphere with the cube.

In the circle, whose semidiameter is AB, inscribe BFGHI the pentagone of a dodecaedron, and BKL the triangle of an Icosaedron. *a* wherefore BG shall be the side of a cube inscribed in the same sphere. therefore BKq *b* $=$ 3 ABq; and Eq *c* $=$ 3 ACq. therefore BKq. Eq *d* :: ABq. ACq *e* :: BGq. BFq. wherefore by inversion BGq. BKq :: BFq. Eq. *f* whence BG.BK :: BF. E. *W.W. to be Dem.* a cor. 17. 13. b 12. 13. c 4. 13. d 15. 5. e 2. 14. f 22. 6.

Prop. VII.

A Dodecaedron is to an Icosaedron, as the side of a Cube is to the side of an Icosaedron, inscribed in one and the same sphere.

Because *a* the same circle comprehends both the pentagone of a dodecaedron, and the triangle of an Icosaedron, *b* the perpendiculars drawn from the center of the sphere to the planes of the pentagone and triangle, shall be equall one to another. Therefore if the dodecaedron and Isocaedron be conceived divided into pyramids, right lines being drawn from the center of the sphere to all the angles, the altitudes of all the pyramids shall be equall one a 3. 14. b 47. 1.

e 5, and 6. 12. to the other. Wherefore since the pyramids *c* of equall height are one to another as their bases, & the superficies of the dodecaedron is equall to twelve pentagones, and the superficies of the Isocaedron to twenty triangles, the dodecaedron shall be to the Icosaedron, as the superficies of the dodecaedron is to the superficies of the Icosaedron, *d* that is, as the side of the cube is to the side of the Icosaedron.

d 5. 14.

PROP. VIII.

The same circle BCDE *comprehends both the square of the cube* BCDE *; and the triangle of the octaedron* FGH *inscribed in one and the same sphere.*

a 15. 13.
b 47. 1
c 14. 13.
d 12. 13.
e 2. def. 3.

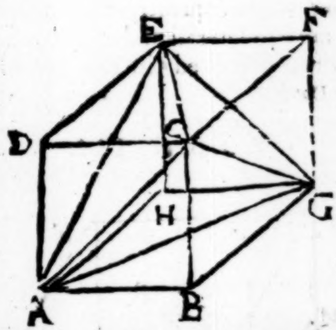
Let A be the diameter of the sphere. Because Aq *a* = 3 BCq *b* = 6 BIq ; and also Aq *c* = 2 GFq *d* = 6 KFq ; thence shall BI be = KF. *e* therefore the circle CBED = GFH. *W.W. to be Demonstrated.*

The End of the fourteenth Book.

THE

THE FIFTEENTH BOOK OF EUCLIDE'S ELEMENTS.

PROPOSITION I.



IN *a cube given* ABGHDCFE *to describe a pyramide* AGEC.

From the angle C draw the diameters CA, CG, CE; and connect them with the diameters AG, GE, EA. All which are [a] equall among themselves, as being the diameters of equall squares: therefore the triangles CAG, CGE, CEA, EAG are equilaterall and equall; and consequently AGEC is a pyramide, which insists upon the angles of the cube, and therefore [b] is inscribed in the same. *W.W. to be Done.* a 47. 1. b 31. def. 11.

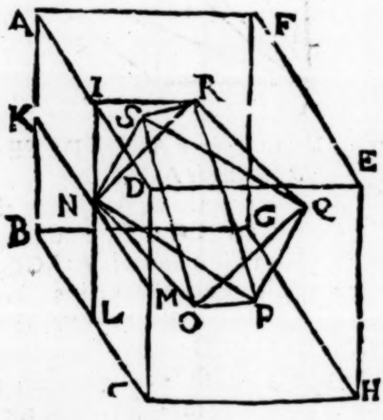
PROP.

PROP. II.

In a pyramide given ABDC *to describe an octaedron* EGKIFH.

a 10. 1. *a* Bisect the sides of the pyramide in the points, E, I, F, K, G, H, which join with the right lines EF, FG, GE,
b 4. 11 &c. All these are *b* equall one to the other; consequently the 8 triangles EHI, IHK, &c. are equilaterall and equall, and so make
c 27. def. 15. *c* an octaedron described *d* in the given pyramide.
d 31. def. 11. *W. W. to be Done.*

PROP. III.



In a cube given CHGBDEFA *to describe an octaedron* NPQSOR.

* 8. 4. Connect* the centers of the squares N, P, Q, S, O, R with the twelve right lines NP, PQ, QS, &c.
a 4. 1. which are *a* equall among themselves; and so make
b 31. and 27. def. 11. 8 equilaterall and equall triangles: wherefore *b* the Octaedron NPQSOR *b* is inscribed in the cube. *W. W. to be Done.*

PROP.

PROP. IV.

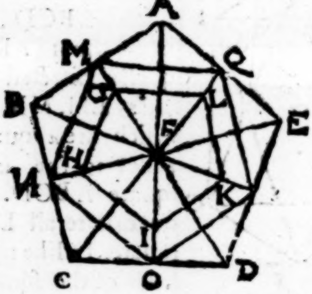
In an Octaedron given ABCDEF, *to inscribe a cube.*

Let the sides of the pyramide EABCD, whose base is the square ABCD, be equally bisected by the right lines, LM, MN, NO, OL, which are *a* equall and *b* parallel to the sides of the square ABCD. *c* then the quadrilaterall LMNO is a square. In like manner, if the sides of the square LMNO be equally bisected in the points G, H, K, I, & GH, HK, KI, IG connected, GHKI shall be a square. And if in the other 5 pyramides of the octaedron, the centers of the triangles be in the same sort conjoined with right lines, then other squares will be described like and equall to the square GHKI. wherefore six such squares shall make a cube, which shall be described within an octaedron, *d* being its eight angles touch the eight bases of the octaedron in their centers. *W.W. to be Done.*

a 4. 1.
b 2. 6.
c 29. def. 1.
d 31. def. 11

PROP.

PROP. V.



In an Icosaedron given to inscribe a Dodecaedron.

Let ABCDEF be a pyramide of the Icosaedron, whose base is the pentagone AB CDE; and the centers of the triangles G, H, I, K, L; which connect with the right lines GH, HI, IK, KL, LG. Then GH-
* 5, 4. IKL shall be a pentagone of the dodecaedron to be inscribed.

For the right lines, FM, FN, FO, FP, FQ, pas-
a cor. 3. 3. sing by the centers of the triangles , *a* do equally
b 4. 1. divide their bases into two parts. *b* therefore the
c 4. 1. right lines MN, NO, OP, PQ, QM *c* are equall one
d 8. 1. to the other; *d* whence also the angles MFN, NFO, OFP , PFQ, QFM are equall. therefore the pentagone GHIKL is equiangular. *e* and consequently
e 4. 1.
f 12. 13. equilaterall, being FG, FH, FI, FK, FL *f* are equall. And if in the other eleven pyramids of the Icosaedron, the centers of the triangles be in like sort conjoined with right lines , then will pentagones equall and like to the pentagone GHIKL be described. Wherefore 12 of such pentagones shall constitute a dodecaedron; which also shall be described in

in the Icosaedron, seeing the twenty angles of the dodecaedron consist upon the centers of the twenty bases of the Icosaedron. Whereby it appears that we have described a dodecaedron in an Isocaedron given. *Which was to be Done.*

F I N I S.